

NONLINEAR OPTICS

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Nicolaas Bloembergen (born 1920) has carried out pioneering studies in nonlinear optics since the early 1960s. He shared the 1981 Nobel Prize with Arthur Schawlow.



Throughout the long history of optics, and indeed until relatively recently, it was thought that all optical media were linear. The consequences of this assumption are far-reaching:

- The optical properties of materials, such as refractive index and absorption coefficient, are independent of light intensity.
- The principle of superposition, a fundamental tenet of classical optics, is applicable.
- The frequency of light is never altered by its passage through a medium.
- Two beams of light in the same region of a medium have no effect on each other so that light cannot be used to control light.

The operation of the first laser in 1960 enabled us to examine the behavior of light in optical materials at higher intensities than previously possible. Experiments carried out in the post-laser era clearly demonstrate that optical media do in fact exhibit nonlinear behavior, as exemplified by the following observations:

- The refractive index, and consequently the speed of light in a nonlinear optical medium, does depend on light intensity.
- The principle of superposition is violated in a nonlinear optical medium.
- The frequency of light is altered as it passes through a nonlinear optical medium; the light can change from red to blue, for example.
- Photons do interact within the confines of a nonlinear optical medium so that light can indeed be used to control light.

The field of nonlinear optics offers a host of fascinating phenomena, many of which are also eminently useful.

Nonlinear optical behavior is not observed when light travels in free space. The “nonlinearity” resides in the medium through which the light travels, rather than in the light itself. The interaction of light with light is therefore mediated by the nonlinear medium: the presence of an optical field modifies the properties of the medium, which in turn causes another optical field, or even the original field itself, to be modified.

As discussed in Chapter 5, the properties of a dielectric medium through which an optical electromagnetic wave propagates are described by the relation between the polarization-density vector $\mathcal{P}(\mathbf{r}, t)$ and the electric-field vector $\mathcal{E}(\mathbf{r}, t)$. Indeed it is useful to view $\mathcal{P}(\mathbf{r}, t)$ as the output of a system whose input is $\mathcal{E}(\mathbf{r}, t)$. The mathematical relation between the vector functions $\mathcal{P}(\mathbf{r}, t)$ and $\mathcal{E}(\mathbf{r}, t)$, which is governed by the characteristics of the medium, defines the system. The medium is said to be nonlinear if this relation is nonlinear (see Sec. 5.2).

This Chapter

In Chapter 5, dielectric media were further classified with respect to their dispersive-ness, homogeneity, and isotropy (see Sec. 5.2). To focus on the principal effect of interest — nonlinearity — the first portion of our exposition is restricted to a medium that is nondispersive, homogeneous, and isotropic. The vectors \mathcal{P} and \mathcal{E} are consequently parallel at every position and time and may therefore be examined on a component-by-component basis.

The theory of nonlinear optics and its applications is presented at two levels. A simplified approach is provided in Secs. 21.1–21.3. This is followed by a more detailed analysis of the same phenomena in Sec. 21.4 and Sec. 21.5.

The propagation of light in media characterized by a second-order (quadratic) nonlinear relation between \mathcal{P} and \mathcal{E} is described in Sec. 21.2 and Sec. 21.4. Applications include the frequency doubling of a monochromatic wave (*second-harmonic generation*), the mixing of two monochromatic waves to generate a third wave at their sum or difference frequencies (*frequency conversion*), the use of two monochromatic waves

to amplify a third wave (*parametric amplification*), and the incorporation of feedback in a parametric-amplification device to create an oscillator (*parametric oscillation*). Wave propagation in a medium with a third-order (cubic) relation between \mathcal{P} and \mathcal{E} is discussed in Secs. 21.3 and 21.5. Applications include *third-harmonic generation*, *self-phase modulation*, *self-focusing*, *four-wave mixing*, and *phase conjugation*. The behavior of anisotropic and dispersive nonlinear optical media is briefly considered in Secs. 21.6 and 21.7, respectively.

Nonlinear Optics in Other Chapters

A principal assumption of the treatment provided in this chapter is that the medium is passive, i.e., it does not exchange energy with the light wave(s). Waves of different frequencies may exchange energy with one another via the nonlinear property of the medium, but their total energy is conserved. This class of nonlinear phenomena are known as **parametric interactions**. Several nonlinear phenomena involving nonparametric interactions are described in other chapters of this book:

- *Laser interactions*. The interaction of light with a medium at frequencies near the resonances of an atomic or molecular transitions involves phenomena such as absorption, and stimulated and spontaneous emission, as described in Sec. 13.3. These interactions become nonlinear when the light is sufficiently intense so that the populations of the various energy levels are significantly altered. Nonlinear optical effects are manifested in the saturation of laser amplifiers and saturable absorbers (Sec. 14.4).
- *Multiphoton absorption*. Intense light can induce the absorption of a collection of photons whose total energy matches that of an atomic transition. For k -photon absorption, the rate of absorption is proportional to I^k , where I is the optical intensity. This nonlinear-optical phenomenon is described briefly in Sec. 13.5B.
- *Nonlinear scattering*. Nonlinear inelastic scattering involves the interaction of light with the vibrational or acoustic modes of a medium. Examples include stimulated Raman and stimulated Brillouin scattering, as described in Secs. 13.5C and 14.3D.

It is also assumed throughout this chapter that the light is described by stationary continuous waves. Nonstationary nonlinear optical phenomena include:

- *Nonlinear optics of pulsed light*. The parametric interaction of optical pulses with a nonlinear medium is described in Sec. 22.5.
- *Optical solitons* are light pulses that travel over exceptionally long distances through nonlinear dispersive media without changing their width or shape. This nonlinear phenomenon is the result of a balance between dispersion and nonlinear self-phase modulation, as described in Sec. 22.5B. The use of solitons in optical fiber communications systems is described in Sec. 24.2E.

Yet another nonlinear optical effect is *optical bistability*. This involves nonlinear optical effects together with feedback. Applications in photonic switching are described in Sec. 23.4.

21.1 NONLINEAR OPTICAL MEDIA

A linear dielectric medium is characterized by a linear relation between the polarization density and the electric field, $\mathcal{P} = \epsilon_o \chi \mathcal{E}$, where ϵ_o is the permittivity of free space and χ is the electric susceptibility of the medium (see Sec. 5.2A). A nonlinear dielectric medium, on the other hand, is characterized by a nonlinear relation between \mathcal{P} and \mathcal{E} (see Sec. 5.2B), as illustrated in Fig. 21.1-1.

The nonlinearity may be of microscopic or macroscopic origin. The polarization

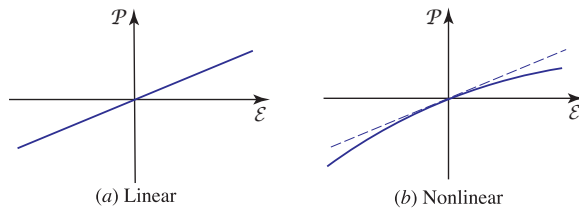


Figure 21.1-1 The \mathcal{P} - \mathcal{E} relation for (a) a linear dielectric medium, and (b) a nonlinear medium.

density $\mathcal{P} = Np$ is a product of the individual dipole moment p induced by the applied electric field \mathcal{E} and the number density of dipole moments N . The nonlinear behavior may reside either in p or in N .

The relation between p and \mathcal{E} is linear when \mathcal{E} is small, but becomes nonlinear when \mathcal{E} acquires values comparable to interatomic electric fields, which are typically $\sim 10^5$ – 10^8 V/m. This may be understood in terms of a simple Lorentz model in which the dipole moment is $p = -ex$, where x is the displacement of a mass with charge $-e$ to which an electric force $-e\mathcal{E}$ is applied (see Sec. 5.5C). If the restraining elastic force is proportional to the displacement (i.e., if Hooke's law is satisfied), the equilibrium displacement x is proportional to \mathcal{E} . In that case \mathcal{P} is proportional to \mathcal{E} and the medium is linear. However, if the restraining force is a nonlinear function of the displacement, the equilibrium displacement x and the polarization density \mathcal{P} are nonlinear functions of \mathcal{E} and, consequently, the medium is nonlinear. The time dynamics of an anharmonic oscillator model describing a dielectric medium with these features is discussed in Sec. 21.7.

Another possible origin of a nonlinear response of an optical material to light is the dependence of the number density N on the optical field. An example is provided by a laser medium in which the number of atoms occupying the energy levels involved in the absorption and emission of light are dependent on the intensity of the light itself (see Sec. 14.4).

Since externally applied optical electric fields are typically small in comparison with characteristic interatomic or crystalline fields, even when focused laser light is used, the nonlinearity is usually weak. The relation between \mathcal{P} and \mathcal{E} is then approximately linear for small \mathcal{E} , deviating only slightly from linearity as \mathcal{E} increases (see Fig. 21.1-1). Under these circumstances, the function that relates \mathcal{P} to \mathcal{E} can be expanded in a Taylor series about $\mathcal{E} = 0$,

$$\mathcal{P} = a_1\mathcal{E} + \frac{1}{2}a_2\mathcal{E}^2 + \frac{1}{6}a_3\mathcal{E}^3 + \cdots, \quad (21.1-1)$$

and it suffices to use only a few terms. The coefficients a_1 , a_2 , and a_3 are the first, second, and third derivatives of \mathcal{P} with respect to \mathcal{E} , evaluated at $\mathcal{E} = 0$. These coefficients are characteristic constants of the medium. The first term, which is linear, dominates at small \mathcal{E} . Clearly, $a_1 = \epsilon_o\chi$, where χ is the linear susceptibility, which is related to the dielectric constant and the refractive index of the material by $n^2 = \epsilon/\epsilon_o = 1 + \chi$ [see (5.2-11)]. The second term represents a quadratic or second-order nonlinearity, the third term represents a third-order nonlinearity, and so on.

It is customary to write (21.1-1) in the form[†]

$$\mathcal{P} = \epsilon_o\chi\mathcal{E} + 2d\mathcal{E}^2 + 4\chi^{(3)}\mathcal{E}^3 + \cdots, \quad (21.1-2)$$

[†] This nomenclature is used in a number of books, such as A. Yariv, *Quantum Electronics*, Wiley, 3rd ed. 1989. An alternative relation, $\mathcal{P} = \epsilon_o(\chi\mathcal{E} + \chi^{(2)}\mathcal{E}^2 + \chi^{(3)}\mathcal{E}^3)$, is used in other books, e.g., Y. R. Shen, *The Principles of Nonlinear Optics*, Wiley, 1984, paperback ed. 2002.

where $d = \frac{1}{4}a_2$ and $\chi^{(3)} = \frac{1}{24}a_3$ are coefficients describing the strength of the second- and third-order nonlinear effects, respectively.

Equation (21.1-2) provides the essential mathematical characterization of a nonlinear optical medium. Material dispersion, inhomogeneity, and anisotropy have not been taken into account both for the sake of simplicity and to enable us to focus on the essential features of nonlinear optical behavior. Sections 21.6 and 21.7 are devoted to anisotropic and dispersive nonlinear media, respectively.

In centrosymmetric media, which have inversion symmetry so that the properties of the medium are not altered by the transformation $\mathbf{r} \rightarrow -\mathbf{r}$, the \mathcal{P} - \mathcal{E} function must have odd symmetry, so that the reversal of \mathcal{E} results in the reversal of \mathcal{P} without any other change. The second-order nonlinear coefficient d must then vanish, and the lowest order nonlinearity is of third order.

Typical values of the second-order nonlinear coefficient d for dielectric crystals, semiconductors, and organic materials used in photonics applications lie in the range $d = 10^{-24}$ – 10^{-21} (C/V² in MKS units). Typical values of the third-order nonlinear coefficient $\chi^{(3)}$ for glasses, crystals, semiconductors, semiconductor-doped glasses, and organic materials of interest in photonics are in the vicinity of $\chi^{(3)} = 10^{-34}$ – 10^{-29} (Cm/V³ in MKS units). Biased or asymmetric quantum wells offer large nonlinearities in the mid and far infrared.

EXERCISE 21.1-1

Intensity of Light Required to Elicit Nonlinear Effects.

- (a) Determine the light intensity (in W/cm²) at which the ratio of the second term to the first term in (21.1-2) is 1% in an ADP (NH₄H₂PO₄) crystal for which $n = 1.5$ and $d = 6.8 \times 10^{-24}$ C/V² at $\lambda_o = 1.06$ μ m.
- (b) Determine the light intensity at which the third term in (21.1-2) is 1% of the first term in carbon disulfide (CS₂) for which $n = 1.6$, $d = 0$, and $\chi^{(3)} = 4.4 \times 10^{-32}$ Cm/V³ at $\lambda_o = 694$ nm.

Note: In accordance with (5.4-8), the light intensity is $I = |\mathcal{E}_0|^2/2\eta = \langle \mathcal{E}^2 \rangle / \eta$, where $\eta = \eta_o/n$ is the impedance of the medium and $\eta_o = (\mu_o/\epsilon_o)^{1/2} \approx 377$ Ω is the impedance of free space (see Sec. 5.4).

The Nonlinear Wave Equation

The propagation of light in a nonlinear medium is governed by the wave equation (5.2-25), which was derived from Maxwell's equations for an arbitrary homogeneous, isotropic dielectric medium. The isotropy of the medium ensures that the vectors \mathcal{P} and \mathcal{E} are always parallel so that they may be examined on a component-by-component basis, which provides

$$\nabla^2 \mathcal{E} - \frac{1}{c_o^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} = \mu_o \frac{\partial^2 \mathcal{P}}{\partial t^2}. \quad (21.1-3)$$

It is convenient to write the polarization density in (21.1-2) as a sum of linear ($\epsilon_o \chi \mathcal{E}$) and nonlinear (\mathcal{P}_{NL}) parts,

$$\mathcal{P} = \epsilon_o \chi \mathcal{E} + \mathcal{P}_{\text{NL}}, \quad (21.1-4)$$

$$\mathcal{P}_{\text{NL}} = 2d\mathcal{E}^2 + 4\chi^{(3)}\mathcal{E}^3 + \dots \quad (21.1-5)$$

Using (21.1-4), along with the relations $c = c_o/n$, $n^2 = 1 + \chi$, and $c_o = 1/(\epsilon_o \mu_o)^{1/2}$ provided in (5.2-11) and (5.2-12), allows (21.1-3) to be written as

$$\nabla^2 \mathcal{E} - \frac{1}{c^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} = -\mathcal{S} \quad (21.1-6)$$

$$\mathcal{S} = -\mu_o \frac{\partial^2 \mathcal{P}_{\text{NL}}}{\partial t^2}. \quad (21.1-7)$$

Wave Equation
in Nonlinear Medium

It is convenient to regard (21.1-6) as a wave equation in which the term \mathcal{S} is regarded as a source that radiates in a linear medium of refractive index n . Because \mathcal{P}_{NL} (and therefore \mathcal{S}) is a nonlinear function of \mathcal{E} , (21.1-6) is a nonlinear partial differential equation in \mathcal{E} . This is the basic equation that underlies the theory of nonlinear optics.

Two approximate approaches to solving this nonlinear wave equation can be called upon. The first is an iterative approach known as the Born approximation. This approximation underlies the simplified introduction to nonlinear optics presented in Secs. 21.2 and 21.3. The second approach is a coupled-wave theory in which the nonlinear wave equation is used to derive linear coupled partial differential equations that govern the interacting waves. This is the basis of the more advanced study of wave interactions in nonlinear media presented in Sec. 21.4 and Sec. 21.5.

Scattering Theory of Nonlinear Optics: The Born Approximation

The radiation source \mathcal{S} in (21.1-6) is a function of the field \mathcal{E} that it, itself, radiates. To emphasize this point we write $\mathcal{S} = \mathcal{S}(\mathcal{E})$ and illustrate the process by the simple block diagram in Fig. 21.1-2. Suppose that an optical field \mathcal{E}_0 is incident on a nonlinear medium confined to some volume as shown in the figure. This field creates a radiation source $\mathcal{S}(\mathcal{E}_0)$ that radiates an optical field \mathcal{E}_1 . The corresponding radiation source $\mathcal{S}(\mathcal{E}_1)$ radiates a field \mathcal{E}_2 , and so on. This process suggests an iterative solution, the first step of which is known as the **first Born approximation**. The second Born approximation carries the process an additional step, and so on. The first Born approximation is

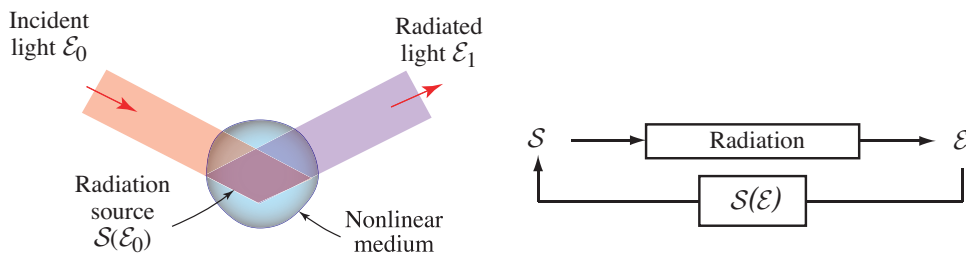


Figure 21.1-2 The first Born approximation. An incident optical field \mathcal{E}_0 creates a source $\mathcal{S}(\mathcal{E}_0)$, which radiates an optical field \mathcal{E}_1 .

adequate when the light intensity is sufficiently weak so that the nonlinearity is small. In this approximation, light propagation through the nonlinear medium is regarded as a scattering process in which the incident field is scattered by the medium. The scattered light is determined from the incident light in two steps:

1. The incident field \mathcal{E}_0 is used to determine the nonlinear polarization density \mathcal{P}_{NL} , from which the radiation source $\mathcal{S}(\mathcal{E}_0)$ is determined.
2. The radiated (scattered) field \mathcal{E}_1 is determined from the radiation source by adding the spherical waves associated with the different source points (as in the theory of diffraction discussed in Sec. 4.3).

The development presented in Sec. 21.2 and Sec. 21.3 are based on the first Born approximation. The initial field \mathcal{E}_0 is assumed to contain one or several monochromatic waves of different frequencies. The corresponding nonlinear polarization \mathcal{P}_{NL} is then determined using (21.1-5) and the source function $\mathcal{S}(\mathcal{E}_0)$ is evaluated using (21.1-7). Since $\mathcal{S}(\mathcal{E}_0)$ is a nonlinear function, new frequencies are created. The source therefore emits an optical field \mathcal{E}_1 with frequencies not present in the original wave \mathcal{E}_0 . This leads to numerous interesting phenomena that have been utilized to make useful nonlinear optics devices.

21.2 SECOND-ORDER NONLINEAR OPTICS

In this section we examine the optical properties of a nonlinear medium in which nonlinearities of order higher than the second are negligible, so that

$$\mathcal{P}_{\text{NL}} = 2d\mathcal{E}^2. \quad (21.2-1)$$

We consider an electric field \mathcal{E} comprising one or two harmonic components and determine the spectral components of \mathcal{P}_{NL} . In accordance with the first Born approximation, the radiation source \mathcal{S} contains the same spectral components as \mathcal{P}_{NL} , and so, therefore, does the emitted (scattered) field.

A. Second-Harmonic Generation (SHG) and Rectification

Consider the response of this nonlinear medium to a harmonic electric field of angular frequency ω (wavelength $\lambda_o = 2\pi c_o/\omega$) and complex amplitude $E(\omega)$,

$$\mathcal{E}(t) = \text{Re}\{E(\omega) \exp(j\omega t)\} = \frac{1}{2}[E(\omega) \exp(j\omega t) + E^*(\omega) \exp(-j\omega t)]. \quad (21.2-2)$$

The corresponding nonlinear polarization density \mathcal{P}_{NL} is obtained by substituting (21.2-2) into (21.2-1),

$$\mathcal{P}_{\text{NL}}(t) = P_{\text{NL}}(0) + \text{Re}\{P_{\text{NL}}(2\omega) \exp(j2\omega t)\} \quad (21.2-3)$$

where

$$P_{\text{NL}}(0) = d E(\omega) E^*(\omega) \quad (21.2-4)$$

$$P_{\text{NL}}(2\omega) = d E^2(\omega). \quad (21.2-5)$$

This process is graphically illustrated in Fig. 21.2-1.

Second-Harmonic Generation (SHG)

The source $\mathcal{S}(t) = -\mu_o \partial^2 \mathcal{P}_{\text{NL}} / \partial t^2$ corresponding to (21.2-3) has a component at frequency 2ω with complex amplitude $S(2\omega) = 4\mu_o \omega^2 d E(\omega) E(\omega)$, which radiates an optical field at frequency 2ω (wavelength $\lambda_o/2$). Thus, the scattered optical field has a component at the second harmonic of the incident optical field. Since the amplitude of the emitted second-harmonic light is proportional to $S(2\omega)$, its intensity $I(2\omega)$ is proportional to $|S(2\omega)|^2$, which is proportional to the square of the intensity of the incident wave $I(\omega) = |E(\omega)|^2/2\eta$ and to the square of the nonlinear coefficient d .

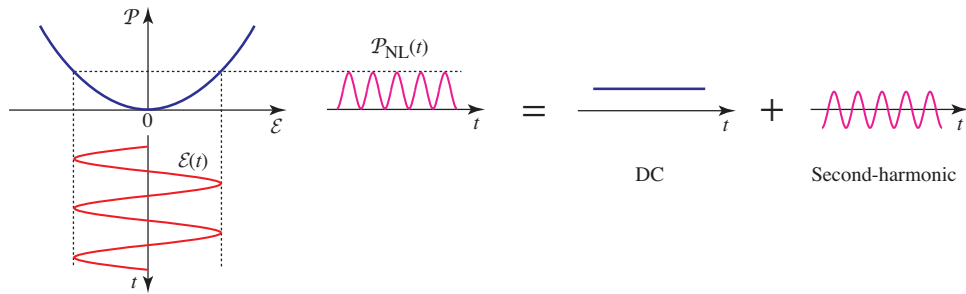


Figure 21.2-1 A sinusoidal electric field of angular frequency ω in a second-order nonlinear optical medium creates a polarization with a component at 2ω (second-harmonic) and a steady (dc) component.

Since the emissions are added coherently, the intensity of the second-harmonic wave is proportional to the square of the length of the interaction volume L .

The efficiency of second-harmonic generation $\eta_{\text{SHG}} = I(2\omega)/I(\omega)$ is therefore proportional to $L^2 I(\omega)$. Since $I(\omega) = P/A$, where P is the incident power and A is the cross-sectional area of the interaction volume, the SHG efficiency is often expressed in the form

$$\eta_{\text{SHG}} = C^2 \frac{L^2}{A} P, \quad (21.2-6)$$

SHG Efficiency

where C^2 is a constant (units of W^{-1}) proportional to d^2 and ω^2 . An expression for C^2 will be provided in (21.4-36).

In accordance with (21.2-6), to maximize the SHG efficiency it is essential that the incident wave have the largest possible power P . This is accomplished by use of pulsed lasers for which the energy is confined in time to obtain large peak powers. Additionally, to maximize the ratio L^2/A , the wave must be focused to the smallest possible area A and provide the longest possible interaction length L . If the dimensions of the nonlinear crystal are not limiting factors, the maximum value of L for a given area A is limited by beam diffraction. For example, a Gaussian beam focused to a beam width W_0 maintains a beam cross-sectional area $A = \pi W_0^2$ over a depth of focus $L = 2z_0 = 2\pi W_0^2/\lambda$ [see (3.1-22)] so that the ratio $L^2/A = 2L/\lambda = 4A/\lambda^2$. The beam should then be focused to the largest spot size, corresponding to the largest depth of focus. In this case, the efficiency is proportional to L . For a thin crystal, L is determined by the crystal and the beam should be focused to the smallest spot area A [see Fig. 21.2-2 (a)]. For a thick crystal, the beam should be focused to the largest spot that fits within the cross-sectional area of the crystal [see Fig. 21.2-2(b)].

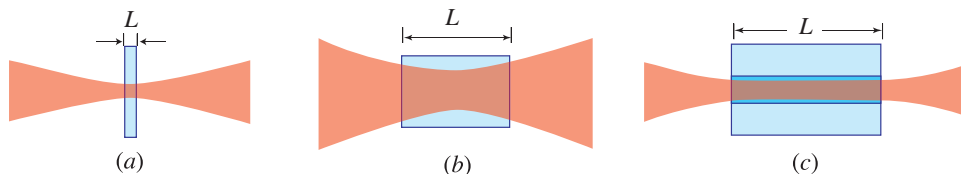


Figure 21.2-2 Interaction volume in a (a) thin crystal, (b) thick crystal, and (c) waveguide.

Guided-wave structures offer the advantage of light confinement in a small cross-sectional area over long distances [see Fig. 21.2-2(c)]. Since A is determined by the size of the guided mode, the efficiency is proportional to L^2 . Optical waveguides take the form of planar or channel waveguides (Chapter 8) or fibers (Chapter 9). Although silica-glass fibers were initially ruled out for second-harmonic generation since glass is centrosymmetric (and therefore presumably has $d = 0$), second-harmonic generation is in fact observed in silica-glass fibers, an effect attributed to electric quadrupole and magnetic dipole interactions and to defects and color centers in the fiber core.

Figure 21.2-3 illustrates several configurations for optical second-harmonic-generation in bulk materials and in waveguides, in which infrared light is converted to visible light and visible light is converted to the ultraviolet.

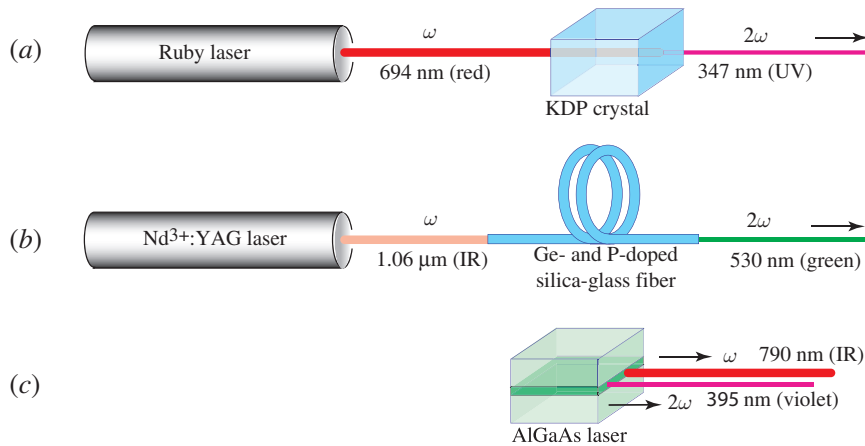


Figure 21.2-3 Optical second-harmonic generation (a) in a bulk crystal; (b) in a glass fiber; (c) within the cavity of a laser diode.

Optical Rectification

The component $P_{\text{NL}}(0)$ in (21.2-3) corresponds to a steady (non-time-varying) polarization density that creates a DC potential difference across the plates of a capacitor within which the nonlinear material is placed (Fig. 21.2-4). The generation of a DC voltage as a result of an intense optical field represents optical rectification (in analogy with the conversion of a sinusoidal AC voltage into a DC voltage in an ordinary electronic rectifier). An optical pulse of several MW peak power, for example, may generate a voltage of several hundred μV .

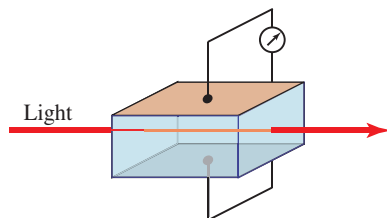


Figure 21.2-4 The transmission of an intense beam of light through a nonlinear crystal generates a DC voltage across it.

B. The Electro-Optic Effect

We now consider an electric field $\mathcal{E}(t)$ comprising a harmonic component at an optical frequency ω together with a steady component (at $\omega = 0$),

$$\mathcal{E}(t) = E(0) + \text{Re}\{E(\omega) \exp(j\omega t)\}. \quad (21.2-7)$$

We distinguish between these two components by denoting the electric field $E(0)$ and the optical field $E(\omega)$. In fact, both components are electric fields.

Substituting (21.2-7) into (21.2-1), we obtain

$$\mathcal{P}_{\text{NL}}(t) = P_{\text{NL}}(0) + \text{Re}\{P_{\text{NL}}(\omega) \exp(j\omega t)\} + \text{Re}\{P_{\text{NL}}(2\omega) \exp(j2\omega t)\}, \quad (21.2-8)$$

where

$$P_{\text{NL}}(0) = d [2E^2(0) + |E(\omega)|^2] \quad (21.2-9a)$$

$$P_{\text{NL}}(\omega) = 4d E(0)E(\omega) \quad (21.2-9b)$$

$$P_{\text{NL}}(2\omega) = d E^2(\omega), \quad (21.2-9c)$$

so that the polarization density contains components at the angular frequencies 0, ω , and 2ω .

If the optical field is substantially smaller in magnitude than the electric field, i.e., $|E(\omega)|^2 \ll |E(0)|^2$, the second-harmonic polarization component $P_{\text{NL}}(2\omega)$ may be neglected in comparison with the components $P_{\text{NL}}(0)$ and $P_{\text{NL}}(\omega)$. This is equivalent to the linearization of \mathcal{P}_{NL} as a function of \mathcal{E} , i.e., approximating it by a straight line with a slope equal to the derivative at $\mathcal{E} = E(0)$, as illustrated in Fig. 21.2-5.

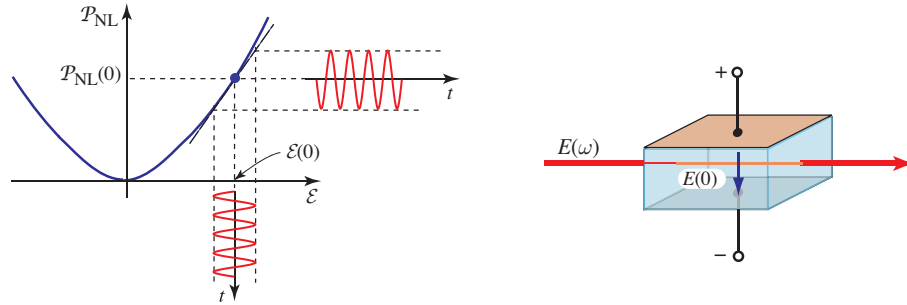


Figure 21.2-5 Linearization of the second-order nonlinear relation $\mathcal{P}_{\text{NL}} = 2d\mathcal{E}^2$ in the presence of a strong electric field $E(0)$ and a weak optical field $E(\omega)$.

Equation (21.2-9b) provides a linear relation between $P_{\text{NL}}(\omega)$ and $E(\omega)$, which we write in the form $P_{\text{NL}}(\omega) = \epsilon_o \Delta\chi E(\omega)$, where $\Delta\chi = (4d/\epsilon_o)E(0)$ represents an increase in the susceptibility proportional to the electric field $E(0)$. The corresponding incremental change of the refractive index is obtained by differentiating the relation $n^2 = 1 + \chi$, to obtain $2n \Delta n = \Delta\chi$, from which

$$\Delta n = \frac{2d}{n\epsilon_o} E(0). \quad (21.2-10)$$

The medium is then effectively linear with a refractive index $n + \Delta n$ that is linearly controlled by the electric field $E(0)$.

The nonlinear nature of the medium creates a coupling between the electric field $E(0)$ and the optical field $E(\omega)$, causing one to control the other, so that the nonlinear medium exhibits the linear electro-optic effect (Pockels effect) discussed in Chapter 20.

This effect is characterized by the relation $\Delta n = -\frac{1}{2}n^3\mathfrak{r}E(0)$, where \mathfrak{r} is the Pockels coefficient. Comparing this formula with (21.2-10), we conclude that the Pockels coefficient \mathfrak{r} is related to the second-order nonlinear coefficient d by

$$\mathfrak{r} \approx -\frac{4}{\epsilon_0 n^4} d. \quad (21.2-11)$$

Although this expression reveals the common underlying origin of the Pockels effect and the medium nonlinearity, it is not consistent with experimentally observed values of \mathfrak{r} and d . This is because we have made the implicit assumption that the medium is nondispersive (i.e., that its response is insensitive to frequency). This assumption is clearly not satisfied when one of the components of the field is at the optical frequency ω and the other is a steady field with zero frequency. The role of dispersion is discussed in Sec. 21.7.

C. Three-Wave Mixing

We now consider the case of a field $\mathcal{E}(t)$ comprising two harmonic components at optical frequencies ω_1 and ω_2 ,

$$\mathcal{E}(t) = \text{Re}\{E(\omega_1) \exp(j\omega_1 t) + E(\omega_2) \exp(j\omega_2 t)\}. \quad (21.2-12)$$

The nonlinear component of the polarization $\mathcal{P}_{\text{NL}} = 2d\mathcal{E}^2$ then contains components at five frequencies, 0 , $2\omega_1$, $2\omega_2$, $\omega_+ = \omega_1 + \omega_2$, and $\omega_- = \omega_1 - \omega_2$, with amplitudes

$$P_{\text{NL}}(0) = d [|E(\omega_1)|^2 + |E(\omega_2)|^2] \quad (21.2-13a)$$

$$P_{\text{NL}}(2\omega_1) = d E(\omega_1)E(\omega_1) \quad (21.2-13b)$$

$$P_{\text{NL}}(2\omega_2) = d E(\omega_2)E(\omega_2) \quad (21.2-13c)$$

$$P_{\text{NL}}(\omega_+) = 2d E(\omega_1)E(\omega_2) \quad (21.2-13d)$$

$$P_{\text{NL}}(\omega_-) = 2d E(\omega_1)E^*(\omega_2). \quad (21.2-13e)$$

Thus, the second-order nonlinear medium can be used to mix two optical waves of different frequencies and generate (among other things) a third wave at the difference frequency or at the sum frequency. The former process is called **frequency downconversion** whereas the latter is known as **frequency up-conversion** or **sum-frequency generation**. An example of frequency up-conversion is provided in Fig. 21.2-6: the light from two lasers with free-space wavelengths $\lambda_{o1} = 1.06 \mu\text{m}$ and $\lambda_{o2} = 10.6 \mu\text{m}$ enter a proustite crystal and generate a third wave with wavelength $\lambda_{o3} = 0.96 \mu\text{m}$ (where $\lambda_{o3}^{-1} = \lambda_{o1}^{-1} + \lambda_{o2}^{-1}$).

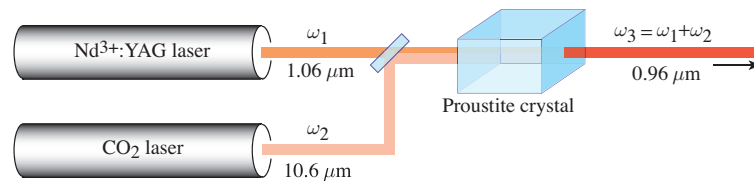


Figure 21.2-6 An example of sum-frequency generation (SFG), also called frequency up-conversion, in a nonlinear crystal.

Although the incident pair of waves at frequencies ω_1 and ω_2 produce polarization densities at frequencies 0 , $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$, and $\omega_1 - \omega_2$, all of these waves are not necessarily generated, since certain additional conditions (phase matching) must be satisfied, as explained presently.

Frequency and Phase Matching

If waves 1 and 2 are plane waves with wavevectors \mathbf{k}_1 and \mathbf{k}_2 , so that $E(\omega_1) = A_1 \exp(-j\mathbf{k}_1 \cdot \mathbf{r})$ and $E(\omega_2) = A_2 \exp(-j\mathbf{k}_2 \cdot \mathbf{r})$, then in accordance with (21.2-13d), $P_{\text{NL}}(\omega_3) = 2dE(\omega_1)E(\omega_2) = 2dA_1A_2 \exp(-j\mathbf{k}_3 \cdot \mathbf{r})$, where

$$\omega_1 + \omega_2 = \omega_3 \quad (21.2-14)$$

Frequency-Matching Condition

and

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3. \quad (21.2-15)$$

Phase-Matching Condition

The medium therefore acts as a light source of frequency $\omega_3 = \omega_1 + \omega_2$, with a complex amplitude proportional to $\exp(-j\mathbf{k}_3 \cdot \mathbf{r})$, so that it radiates a wave of wavevector $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$, as illustrated in Fig. 21.2-7. Equation (21.2-15) can be regarded as a condition of phase matching among the wavefronts of the three waves that is analogous to the frequency-matching condition $\omega_1 + \omega_2 = \omega_3$. Since the argument of the complex wavefunction is $\omega t - \mathbf{k} \cdot \mathbf{r}$, these two conditions ensure both the temporal and spatial phase matching of the three waves, which is necessary for their sustained mutual interaction over extended durations of time and regions of space.

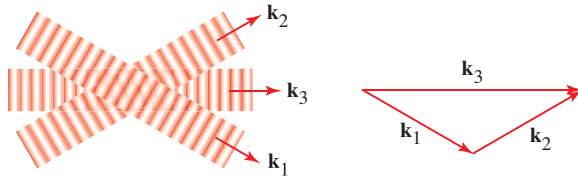


Figure 21.2-7 The phase-matching condition.

Three-Wave Mixing Modalities

When two optical waves of angular frequencies ω_1 and ω_2 travel through a second-order nonlinear optical medium they mix and produce a polarization density with components at a number of frequencies. We assume that only the component at the sum frequency $\omega_3 = \omega_1 + \omega_2$ satisfies the phase-matching condition. Other frequencies cannot be sustained by the medium since they are assumed not to satisfy the phase-matching condition.

Once wave 3 is generated, it interacts with wave 1 and generates a wave at the difference frequency $\omega_2 = \omega_3 - \omega_1$. Clearly, the phase-matching condition for this interaction is also satisfied. Waves 3 and 2 similarly combine and radiate at ω_1 . The three waves therefore undergo mutual coupling in which each pair of waves interacts and contributes to the third wave. The process is called **three-wave mixing**.

Two-wave mixing is not, in general, possible. Two waves of arbitrary frequencies ω_1 and ω_2 cannot be coupled by the medium without the help of a third wave. Two-wave mixing can occur only in the degenerate case, $\omega_2 = 2\omega_1$, in which the second-harmonic of wave 1 contributes to wave 2; and the subharmonic $\omega_2/2$ of wave 2, which is at the frequency difference $\omega_2 - \omega_1$, contributes to wave 1.

Three-wave mixing is known as a **parametric interaction** process. It takes a variety of forms, depending on which of the three waves is provided as an input, and which are extracted as outputs, as illustrated in the following examples (see Fig. 21.2-8):

- **Optical Frequency Conversion (OFC).** Waves 1 and 2 are mixed in an **up-converter**, generating a wave at the sum frequency $\omega_3 = \omega_1 + \omega_2$. This process, also called **sum-frequency generation (SFG)**, has already been illustrated in Fig. 21.2-6. Second-harmonic generation (SHG) is a degenerate special case of SFG. The opposite process of **downconversion** or **frequency-difference generation** is realized by an interaction between waves 3 and 1 to generate wave 2, at the difference frequency $\omega_2 = \omega_3 - \omega_1$. Up- and down-converters are used to generate coherent light at wavelengths where no adequate lasers are available, and as optical mixers in optical communication systems.
- **Optical Parametric Amplifier (OPA).** Waves 1 and 3 interact so that wave 1 grows, and in the process an auxiliary wave 2 is created. The device operates as a coherent amplifier at frequency ω_1 and is known as an OPA. Wave 3, called the **pump**, provides the required energy, whereas wave 2 is known as the **idler** wave. The amplified wave is called the **signal**. Clearly, the gain of the amplifier depends on the power of the pump. OPAs are used for the detection of weak light at wavelengths for which sensitive detectors are not available.
- **Optical Parametric Oscillator (OPO).** With proper feedback, the parametric amplifier can operate as a parametric oscillator, in which only a pump wave is supplied. OPOs are used for the generation of coherent light and mode-locked pulse trains over a continuous range of frequencies, usually in frequency bands where there is a paucity of tunable laser sources.
- **Spontaneous Parametric Downconversion (SPDC).** Here, the only input to the nonlinear crystal is the pump wave 3, and downconversion to the lower-frequency waves 2 and 3 is spontaneous. The frequency- and phase-matching conditions (21.2-14) and (21.2-15) lead to multiple solutions, each forming a pair of waves 1 and 2 with specific frequencies and directions. The down-converted light takes the form of a cone of multispectral light, as illustrated in Fig. 21.2-8.

Further details pertaining to these parametric devices are provided in Sec. 21.4.

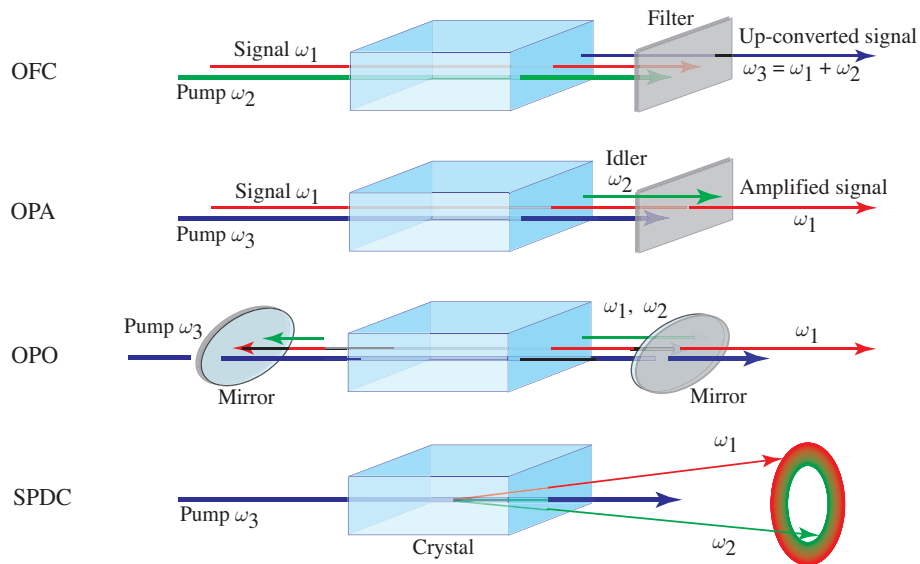


Figure 21.2-8 Optical parametric devices: optical frequency converter (OFC); optical parametric amplifier (OPA); optical parametric oscillator (OPO); spontaneous parametric down-converter (SPDC).

Wave Mixing as a Photon Interaction Process

The three-wave-mixing process can be viewed from a photon-optics perspective as a process of three-photon interaction in which two photons of lower frequency, ω_1 and ω_2 , are annihilated, and a photon of higher frequency ω_3 is created, as illustrated in Fig. 21.2-9(a). Alternatively, the annihilation of a photon of high frequency ω_3 is accompanied by the creation of two low-frequency photons, of frequencies ω_1 and ω_2 , as illustrated in Fig. 21.2-9(b). Since $\hbar\omega$ and $\hbar\mathbf{k}$ are the energy and momentum of a photon of frequency ω and wavevector \mathbf{k} (see Sec. 12.1), conservation of energy and momentum, in either case, requires that

$$\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3 \quad (21.2-16)$$

$$\hbar\mathbf{k}_1 + \hbar\mathbf{k}_2 = \hbar\mathbf{k}_3, \quad (21.2-17)$$

where \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{k}_3 are the wavevectors of the three photons. The frequency- and phase-matching conditions presented in (21.2-14) and (21.2-15) are therefore reproduced.

The energy diagram for the three-photon-mixing process displayed in Fig. 21.2-9(b) bears some similarity to that for an optically pumped three-level laser, illustrated in Fig. 21.2-9(c) (see Sec. 14.2B). There are significant distinctions between the two processes, however:

- One of the three transitions involved in the laser process is non-radiative.
- An exchange of energy between the field and medium takes place in the laser process.
- The energy levels associated with the laser process are relatively sharp and are established by the atomic or molecular system, whereas the energy levels of the parametric process are dictated by photon energy and phase-matching conditions and are tunable over wide spectral regions.

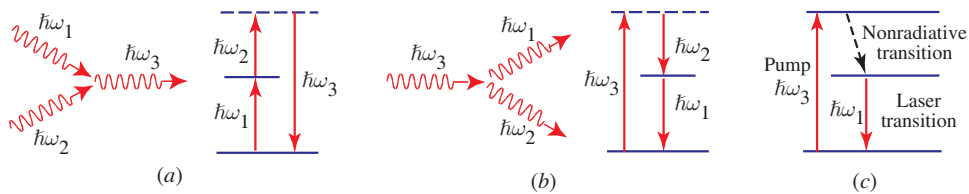


Figure 21.2-9 Comparison of parametric processes in a second-order nonlinear medium and laser action. (a) Annihilation of two low-frequency photons and creation of one high-frequency photon. The dashed line for the upper level indicates that it is virtual. (b) Annihilation of one high-frequency photon and creation of two low-frequency photons. (c) Optically pumped 3-level laser, a nonparametric process in which the medium participates in energy transfer.

The process of wave mixing involves an energy exchange among the interacting waves. Clearly, energy must be conserved, as is assured by the frequency-matching condition, $\omega_1 + \omega_2 = \omega_3$. Photon numbers must also be conserved, consistent with the photon interaction. Consider the photon-splitting process represented in Fig. 21.2-9(b). If $\Delta\phi_1$, $\Delta\phi_2$, and $\Delta\phi_3$ are the net changes in the photon fluxes (photons per second) in the course of the interaction (the flux of photons leaving minus the flux of photons entering) at frequencies ω_1 , ω_2 , and ω_3 , then $\Delta\phi_1 = \Delta\phi_2 = -\Delta\phi_3$, so that for each of the ω_3 photons lost, one each of the ω_1 and ω_2 photons is gained.

If the three waves travel in the same direction, the z direction for example, then by taking a cylinder of unit area and incremental length $\Delta z \rightarrow 0$ as the interaction

volume, we conclude that the photon flux densities ϕ_1 , ϕ_2 , ϕ_3 (photons/s-m²) of the three waves must satisfy

$$\boxed{\frac{d\phi_1}{dz} = \frac{d\phi_2}{dz} = -\frac{d\phi_3}{dz}} \quad (21.2-18)$$

Photon-Number Conservation

Since the wave intensities (W/m²) are $I_1 = \hbar\omega_1\phi_1$, $I_2 = \hbar\omega_2\phi_2$, and $I_3 = \hbar\omega_3\phi_3$, (21.2-18) gives

$$\boxed{\frac{d}{dz} \left(\frac{I_1}{\omega_1} \right) = \frac{d}{dz} \left(\frac{I_2}{\omega_2} \right) = -\frac{d}{dz} \left(\frac{I_3}{\omega_3} \right)} \quad (21.2-19)$$

Manley–Rowe Relation

Equation (21.2-19) is known as the Manley–Rowe relation. It was derived in the context of wave interactions in nonlinear electronic systems. The Manley–Rowe relation can be derived using wave optics, without invoking the concept of the photon (see Exercise 21.4-2).

D. Phase Matching and Tuning Curves

Phase Matching in Collinear Three-Wave Mixing

If the mixed three waves are collinear, i.e., they travel in the same direction, and if the medium is nondispersive, then the phase-matching condition (21.2-15) yields the scalar equation $n\omega_1/c_o + n\omega_2/c_o = n\omega_3/c_o$, which is automatically satisfied if the frequency matching condition $\omega_1 + \omega_2 = \omega_3$ is met. However, since all materials are in reality dispersive, the three waves actually travel at different velocities corresponding to different refractive indexes, n_1 , n_2 , and n_3 , and the frequency- and phase-matching conditions are independent:

$$\boxed{\omega_1 + \omega_2 = \omega_3, \quad \omega_1 n_1 + \omega_2 n_2 = \omega_3 n_3,} \quad (21.2-20)$$

Matching Conditions

and must be simultaneously satisfied. Since this is usually not possible, birefringence, which is present in anisotropic media, is often used to compensate dispersion.

For an anisotropic medium, the three refractive indexes n_1 , n_2 , and n_3 are generally dependent on the polarization of the waves and their directions relative to the principal axes (see Sec. 6.3C). This offers other degrees of freedom to satisfy the matching conditions. Precise control of the refractive indexes at the three frequencies is often achieved by appropriate selection of polarization, orientation of the crystal, and in some cases by control of the temperature.

In practice, the medium is often a uniaxial crystal characterized by its optic axis and frequency-dependent ordinary and extraordinary refractive indexes $n_o(\omega)$ and $n_e(\omega)$. Each of the three waves can be ordinary (o) or extraordinary (e) and the process is labeled accordingly. For example, the label e-o-o indicates that waves 1, 2, and 3 are e, o, and o waves, respectively. For an o wave, $n(\omega) = n_o(\omega)$; for an e wave, $n(\omega) = n(\theta, \omega)$ depends on the angle θ between the direction of the wave and the optic axis of the crystal, in accordance with the relation

$$\frac{1}{n^2(\theta, \omega)} = \frac{\cos^2 \theta}{n_o^2(\omega)} + \frac{\sin^2 \theta}{n_e^2(\omega)}, \quad (21.2-21)$$

which is represented graphically by an ellipse [see (6.3-15) and Fig. 6.3-7]. If the polarizations of the signal and idler waves are the same, the wave mixing is said to be **Type I**; if they are orthogonal, it is said to be **Type II**.

EXAMPLE 21.2-1. Collinear Type-I Second-Harmonic Generation (SHG). For SHG, waves 1 and 2 have the same frequency ($\omega_1 = \omega_2 = \omega$) and $\omega_3 = 2\omega$. For Type-I mixing, waves 1 and 2 have identical polarization so that $n_1 = n_2$. Therefore, from (21.2-20), the phase-matching condition is $n_3 = n_1$, i.e., the fundamental wave has the same refractive index as the second-harmonic wave. Because of dispersion, this condition cannot usually be satisfied unless the polarization of these two waves is different. For a uniaxial crystal, the process is either o-o-e or e-e-o. In either case, the direction at which the wave enters the crystal is adjusted in such a way that $n_3 = n_1$, i.e., such that birefringence compensates exactly for dispersion.

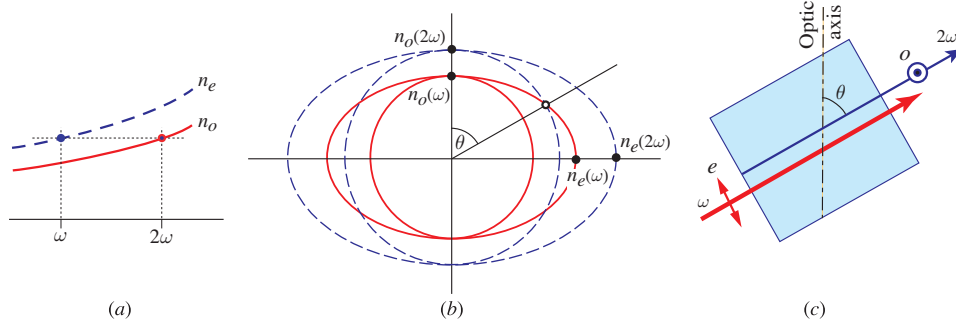


Figure 21.2-10 Phase matching in e-e-o SHG. (a) Matching the index of the e wave at ω with that of the o wave at 2ω . (b) Index surfaces at ω (solid curves) and 2ω (dashed curves) for a uniaxial crystal. (c) The wave is chosen to travel at an angle θ with respect to the crystal optic axis, such that the extraordinary refractive index $n_e(\theta, \omega)$ of the ω wave equals the ordinary refractive index $n_o(2\omega)$ of the 2ω wave.

For an e-e-o process such as that illustrated in Fig. 21.2-10, the fundamental wave is extraordinary and the second-harmonic wave is ordinary, $n_1 = n(\theta, \omega)$ and $n_3 = n_o(2\omega)$, so that the matching condition is: $n(\theta, \omega) = n_o(2\omega)$. This is achieved by selecting an angle θ for which

$$n(\theta, \omega) = n_o(2\omega), \quad (21.2-22)$$

SHG Type-I e-e-o

where $n(\theta, \omega)$ is given by (21.2-21). This is illustrated graphically in Fig. 21.2-10, which displays the ordinary and extraordinary refractive indexes (a circle and an ellipse) at ω (solid curves) and at 2ω (dashed curves). The angle at which phase matching is satisfied is that at which the circle at 2ω intersects the ellipse at ω .

As an example, for KDP at a fundamental wavelength $\lambda = 694$ nm, $n_o(\omega) = 1.506$, $n_e(\omega) = 1.466$; and at $\lambda/2 = 347$ nm, $n_o(2\omega) = 1.534$, $n_e(2\omega) = 1.490$. In this case, (21.2-22) and (21.2-21) gives $\theta = 52^\circ$. This is called the cut angle of the crystal. Similar equations may be written for SHG in the o-o-e configuration. In this case, for KDP at a fundamental wavelength $\lambda = 1.06$ μm , $\theta = 41^\circ$.

EXAMPLE 21.2-2. Collinear Optical Parametric Oscillator (OPO). The oscillation frequencies of an OPO are determined from the frequency and phase matching conditions. For a Type-I o-o-e mixing configuration,

$$\omega_1 + \omega_2 = \omega_3, \quad \omega_1 n_o(\omega_1) + \omega_2 n_o(\omega_2) = \omega_3 n(\theta, \omega_3). \quad (21.2-23)$$

OPO Type-I o-o-e

For Type-II e-o-e mixing,

$$\omega_1 + \omega_2 = \omega_3, \quad \omega_1 n(\theta, \omega_1) + \omega_2 n_o(\omega_2) = \omega_3 n(\theta, \omega_3). \quad (21.2-24)$$

OPO Type-II e-o-e

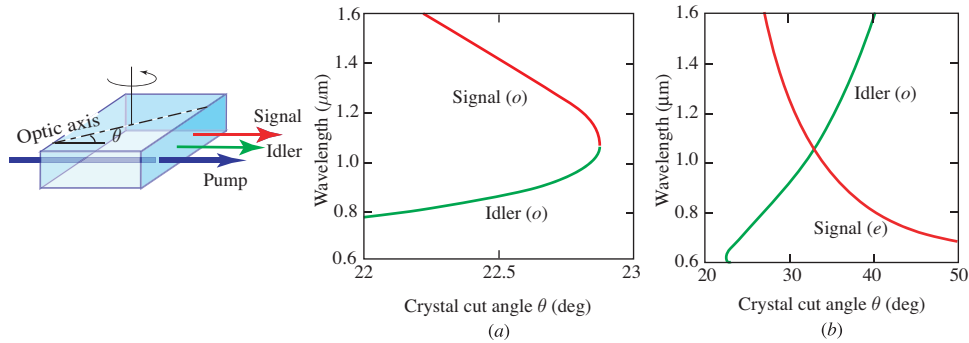


Figure 21.2-11 Tuning curves for a collinear OPO using a BBO crystal and a 532-nm pump, which is readily obtained from a frequency doubled Nd:YAG laser (a) Type I, and (b) Type II.

The functions $n_o(\omega)$ and $n_e(\omega)$ are determined from the Sellmeier equation (5.5-28), and the extraordinary index $n(\theta, \omega)$ is determined as a function of the angle θ between the optic axis of the crystal and the direction of the waves by use of (21.2-21). For a given pump frequency ω_3 , the solutions of (21.2-23) and (21.2-24), ω_1 and ω_2 , are often plotted versus the angle θ , a plot known as the tuning curve. Examples are illustrated in Fig. 21.2-11.

Phase Matching in Non-Collinear Three-Wave Mixing

In the non-collinear case, the phase-matching condition $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ is equivalent to $\omega_1 n_1 \hat{u}_1 + \omega_2 n_2 \hat{u}_2 = \omega_3 n_3 \hat{u}_3$, where \hat{u}_1 , \hat{u}_2 , and \hat{u}_3 are unit vectors in the directions of propagation of the waves. The refractive indexes n_1 , n_2 , and n_3 depend on the directions of the waves relative to the crystal axes, as well as the polarization and frequency. This vector equation is equivalent to two scalar equations so that the matching conditions become

$$\omega_1 + \omega_2 = \omega_3, \quad \omega_1 n_1 \sin \theta_1 = \omega_2 n_2 \sin \theta_2, \quad \omega_1 n_1 \cos \theta_1 + \omega_2 n_2 \cos \theta_2 = \omega_3 n_3, \quad (21.2-25)$$

where θ_1 and θ_2 are the angles waves 1 and 2 make with wave 3. The design of a 3-wave mixing device centers about the selection of directions and polarizations to satisfy these equations, as demonstrated by the following exercises and examples.

EXERCISE 21.2-1

Non-Collinear Type-II Second-Harmonic Generation (SHG). Figure 21.2-12 illustrates Type-II o-e-e non-collinear SHG. An ordinary wave and an extraordinary wave, both at the fundamental frequency ω , create an extraordinary second-harmonic wave at the frequency 2ω . It is assumed here that the directions of propagation of the three waves and the optic axis are coplanar and the two fundamental waves and the optic axis make angles θ_1 , θ_2 , and θ with the direction of the second-harmonic wave. The refractive indexes that appear in the phase-matching equations (21.2-25) are $n_1 = n_o(\omega)$, $n_2 = n(\theta + \theta_2, \omega)$, and $n_3 = n(\theta, 2\omega)$, i.e.,

$$n_o(\omega) \sin \theta_1 = n(\theta + \theta_2, \omega) \sin \theta_2, \quad n_o(\omega) \cos \theta_1 + n(\theta + \theta_2, \omega) \cos \theta_2 = 2n(\theta, 2\omega). \quad (21.2-26)$$

SHG Type-II o-e-e

For a KDP crystal and a fundamental wave of wavelength $1.06\mu\text{m}$ (Nd:Yag laser), determine the crystal orientation and the angles θ_1 and θ_2 for efficient second-harmonic generation.

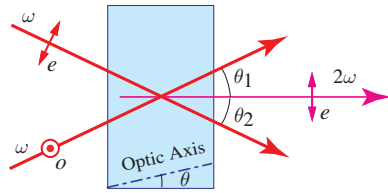


Figure 21.2-12 Non-collinear Type II second-harmonic generation.

EXAMPLE 21.2-3. Spontaneous Parametric Downconversion (SPDC). In SPDC, a pump wave of frequency ω_3 creates pairs of waves 1 and 2, at frequencies ω_1 and ω_2 , and angles θ_1 and θ_2 , all satisfying the frequency- and phase-matching conditions (21.2-25). For example, in the Type-I o-o-e case, $n_1 = n_o(\omega_1)$, $n_2 = n_o(\omega_2)$ and $n_3 = n_e(\theta, \omega_3)$. These relations together with the Sellmeier equations for $n_o(\omega)$ and $n_e(\omega)$ yield a continuum of solutions (ω_1, θ_1) , (ω_2, θ_2) for the signal and idler waves, as illustrated by the example in Fig. 21.2-13.

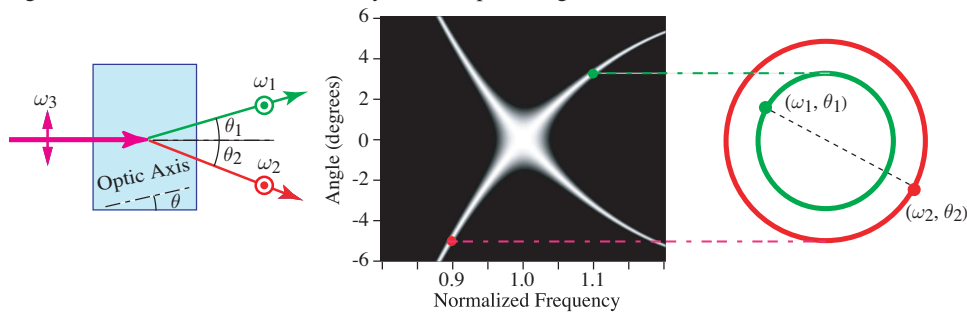


Figure 21.2-13 Tuning curves for non-collinear Type-I o-o-e spontaneous parametric downconversion in a BBO crystal at an angle $\theta = 33.53^\circ$ for a 351.5-nm pump (from an Ar^+ -ion laser). Each point in the bright area of the middle picture represents the frequency ω_1 and angle θ_1 of a possible down-converted wave, and has a matching point at a complementary frequency $\omega_2 = \omega_3 - \omega_1$ with angle θ_2 . Frequencies are normalized to the degenerate frequency $\omega_o = \omega_3/2$. For example, the two dots shown represent a pair of down-converted waves at frequencies $0.9\omega_o$ and $1.1\omega_o$. Because of circular symmetry, each point is actually a ring of points all of the same frequency, but each point on a ring matches only one diametrically opposite point on the corresponding ring, as illustrated in the right graph.

Tolerable Phase Mismatch and Coherence Length

A slight phase mismatch $\Delta\mathbf{k} = \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2 \neq 0$ may result in a significant reduction in the wave-mixing efficiency. If waves 1 and 2 are plane waves with wavevectors \mathbf{k}_1 and \mathbf{k}_2 , so that $E(\omega_1) = A_1 \exp(-j\mathbf{k}_1 \cdot \mathbf{r})$ and $E(\omega_2) = A_2 \exp(-j\mathbf{k}_2 \cdot \mathbf{r})$, then in accordance with (21.2-13d), $P_{\text{NL}}(\omega_3) = 2dE(\omega_1)E(\omega_2) = 2dA_1A_2 \exp[-j(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}] = 2dA_1A_2 \exp(j\Delta\mathbf{k} \cdot \mathbf{r}) \exp(-j\mathbf{k}_3 \cdot \mathbf{r})$. By virtue of (21.1-7) this creates a source with angular frequency ω_3 , wavevector \mathbf{k}_3 , and complex amplitude $2\omega_3^2\mu_o dA_1A_2 \exp(j\Delta\mathbf{k} \cdot \mathbf{r})$. It can be shown (see Prob. 21.2-6) that the intensity of the generated wave is proportional to the squared integral of the source amplitude over the interaction volume V ,

$$I_3 \propto \left| \int_V dA_1A_2 \exp(j\Delta\mathbf{k} \cdot \mathbf{r}) d\mathbf{r} \right|^2. \tag{21.2-27}$$

Because the contributions of different points within the interaction volume are added as phasors, the position-dependent phase $\Delta \mathbf{k} \cdot \mathbf{r}$ in the phase mismatched case results in a reduction of the total intensity below the value obtained in the matched case. Consider the special case of a one-dimensional interaction volume of width L in the z direction: $I_3 \propto |\int_0^L \exp(j\Delta k z) dz|^2 = L^2 \text{sinc}^2(\Delta k L/2\pi)$, where Δk is the z component of $\Delta \mathbf{k}$ and $\text{sinc}(x) = \sin(\pi x)/(\pi x)$. It follows that in the presence of a wavevector mismatch Δk , I_3 is reduced by the factor $\text{sinc}^2(\Delta k L/2\pi)$, which is unity for $\Delta k = 0$ and drops as Δk increases, reaching a value of $(2/\pi)^2 \approx 0.4$ when $|\Delta k| = \pi/L$, and vanishing when $|\Delta k| = 2\pi/L$ (see Fig. 21.2-14). For a given L , the mismatch Δk corresponding to a prescribed efficiency reduction factor is inversely proportional to L , so that the phase-matching requirement becomes more stringent as L increases. For a given mismatch Δk , the length

$$L_c = 2\pi/|\Delta k|$$

(21.2-28)
Coherence Length

is a measure of the maximum length within which the parametric interaction process is efficient; L_c is often called the wave-mixing **coherence length**.

For example, for a second-harmonic generation $|\Delta k| = 2(2\pi/\lambda_o)|n_3 - n_1|$, where λ_o is the free-space wavelength of the fundamental wave and n_1 and n_3 are the refractive indexes of the fundamental and the second-harmonic waves. In this case, $L_c = \lambda_o/2|n_3 - n_1|$ is inversely proportional to $|n_3 - n_1|$, which is governed by the material dispersion. For example, for $|n_3 - n_1| = 10^{-2}$, $L_c = 50\lambda$.

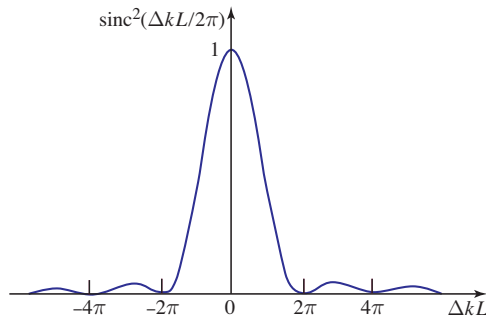


Figure 21.2-14 The factor by which the efficiency of three-wave mixing is reduced as a result of a phase mismatch ΔkL between waves interacting within a distance L .

The tolerance of the interaction process to the phase mismatch can be regarded as a result of the wavevector uncertainty $\Delta k \propto 1/L$ associated with confinement of the waves within a distance L [see (A.2-6) in Appendix A]. The corresponding momentum uncertainty $\Delta p = \hbar\Delta k \propto 1/L$ explains the apparent violation of the law of conservation of momentum in the wave-mixing process.

Phase-Matching Bandwidth

As previously noted, for a finite interaction length L , a phase mismatch $|\Delta k| \leq 2\pi/L$ is tolerated. If exact phase matching is achieved at a set of nominal frequencies of the mixed waves, then small frequency deviations from those values may be tolerated, as long as the condition $\omega_1 + \omega_2 = \omega_3$ is perfectly satisfied. The spectral bands associated with such tolerance are established by the condition $|\Delta k| \leq 2\pi/L$.

As an example, in SHG we have two waves with frequencies $\omega_1 = \omega$ and $\omega_3 = 2\omega$. The mismatch Δk is a function $\Delta k(\omega)$ of the fundamental frequency ω . The device is designed for exact phase matching at a nominal fundamental frequency ω_0 , i.e.,

$\Delta k(\omega_0) = 0$. The bandwidth $\Delta\omega$ is then established by the condition $|\Delta k(\omega_0 + \Delta\omega)| = 2\pi/L$. If $\Delta\omega$ is sufficiently small, we may write $\Delta k(\omega_0 + \Delta\omega) = \Delta k' \Delta\omega$, where $\Delta k' = (d/d\omega)\Delta k$ at ω_0 . Therefore, $\Delta\omega = 2\pi/|\Delta k'|L$, from which the spectral width in Hz is

$$\Delta\nu = 1/|\Delta k'|L. \quad (21.2-29)$$

Phase-Matching Bandwidth

Since $\Delta k(\omega) = k_3(2\omega) - 2k_1(\omega)$, the derivative $\Delta k' = dk_3(2\omega)/d\omega - 2dk_1(\omega)/d\omega = 2[dk_3(2\omega)/d(2\omega) - dk_1(\omega)/d\omega] = 2[1/v_3 - 1/v_1]$, where v_1 and v_3 are the group velocities of waves 1 and 3 at frequencies ω and 2ω , respectively (see Sec. 5.6). The spectral width is therefore related to the length L and the group velocity mismatch by

$$\Delta\nu = \frac{1}{2} \left| \frac{L}{v_3} - \frac{L}{v_1} \right|^{-1} = \frac{c_0}{2L} \frac{1}{|N_3 - N_1|}, \quad (21.2-30)$$

Phase-Matching Bandwidth

where N_1 and N_3 are the group indexes of the material at the fundamental and second-harmonic frequencies.

It is apparent that second-harmonic generation of a broadband wave, or an ultranarrow pulse (see Sec. 23.5), can be accomplished by use of a thin crystal (at a cost of lower conversion efficiency), and by the use of an additional design constraint, group velocity matching, $v_3 \approx v_1$ or $N_3 \approx N_1$. Phase-matching tolerance in SPDC is revealed in Fig. 21.2-13 by the thickness of the curves.

E. Quasi-Phase Matching

In the presence of a wavevector mismatch $\Delta\mathbf{k}$, points within the interaction volume radiate with position-dependent phases $\Delta\mathbf{k} \cdot \mathbf{r}$, so that the magnitude of the generated parametric wave is significantly reduced. Since phase matching can be difficult to achieve, or can severely constrain the choice of the nonlinear coefficient or the crystal configuration that maximizes the efficiency of wave conversion, one approach is to allow a phase mismatch, but to compensate it by using a medium with position-dependent periodic nonlinearity. Such periodicity introduces an opposite phase that brings back the phases of the distributed radiation elements into better alignment. The technique is called **quasi-phase matching (QPM)**.

If the medium has a position-dependent nonlinear coefficient $d(\mathbf{r})$, then (21.2-27) becomes

$$I_3 \propto \left| \int_V d(\mathbf{r}) \exp(j\Delta\mathbf{k} \cdot \mathbf{r}) d\mathbf{r} \right|^2. \quad (21.2-31)$$

If $d(\mathbf{r})$ is a harmonic function $d(\mathbf{r}) = d_0 \exp(-j\mathbf{G} \cdot \mathbf{r})$, with $\mathbf{G} = \Delta\mathbf{k}$, then the phase mismatch is fully eliminated. Accordingly, the phase-matching condition (21.2-15) is replaced with

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{G} = \mathbf{k}_3. \quad (21.2-32)$$

In effect, the nonlinear medium serves as a phase grating (or longitudinal Bragg grating) with a wavevector \mathbf{G} .

It is generally difficult to fabricate a medium with a continuously varying harmonic nonlinear coefficient, $d(\mathbf{r}) = d_o \exp(-j\mathbf{G} \cdot \mathbf{r})$, but it is possible to fabricate simpler periodic structures, e.g., media with nonlinear coefficients of constant magnitude but periodically reversed sign. Since any periodic function can be decomposed into a superposition of harmonic functions via Fourier series, one such function can serve to correct the phase mismatch, with the others playing no role in the wave-mixing process because they introduce greater phase mismatch.

QPM in Collinear Wave Mixing

For collinear waves traveling in the z direction and having a phase mismatch Δk , the required phase grating is of the form $\exp(-jGz)$, where $G = \Delta k$. Such grating may be obtained by use of a periodic nonlinear coefficient $d(z)$ described by the Fourier series $d(z) = \sum_{m=-\infty}^{\infty} d_m \exp(-j2\pi mz/\Lambda)$, where Λ is the period and $\{d_m\}$ are the Fourier coefficients. Any of these components may be used for phase matching. For example, for the m th harmonic, $G = m2\pi/\Lambda = \Delta k$, so that

$$\Lambda = m2\pi/\Delta k = mL_c, \quad (21.2-33)$$

QPM Condition

i.e., the grating period Λ equals an integer multiple of the coherence length $L_c = 2\pi/\Delta k$.

Equation (21.2-32) together with the frequency matching condition yield

$$\omega_1 + \omega_2 = \omega_3, \quad \omega_1 n_1 + \omega_2 n_2 + m2\pi c/\Lambda = \omega_3 n_3. \quad (21.2-34)$$

QPM Tuning Curves

These equations are used in lieu of (21.2-20) to determine the tuning curves and the crystal angles in the design of parametric devices. It is evident that QPM offers some flexibility in the design of desired tuning curves.

QPM in a Medium with Periodically Reversed Nonlinear Coefficient

The simplest periodic pattern of the nonlinear coefficient $d(z)$ alternates between two constant values, $+d_o$ and $-d_o$, at distances $\Lambda/2$, as illustrated in Fig. 21.2-15.

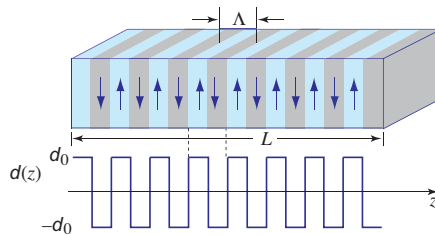


Figure 21.2-15 A nonlinear crystal with periodically varying nonlinear coefficient $d(z)$ of period Λ .

The physical mechanism by which the periodic reversal of the sign of nonlinearity serves to compensate the position-dependent phase of the radiation is illustrated in Fig. 21.2-16 in the $m = 1$ case; i.e., the grating period Λ equals the coherence length $L_c = 2\pi/\Delta k$.

The improvement of the conversion efficiency afforded by QPM may be determined quantitatively as follows. In accordance with Fourier series theory, $d_m = (2/m\pi)d_o$,

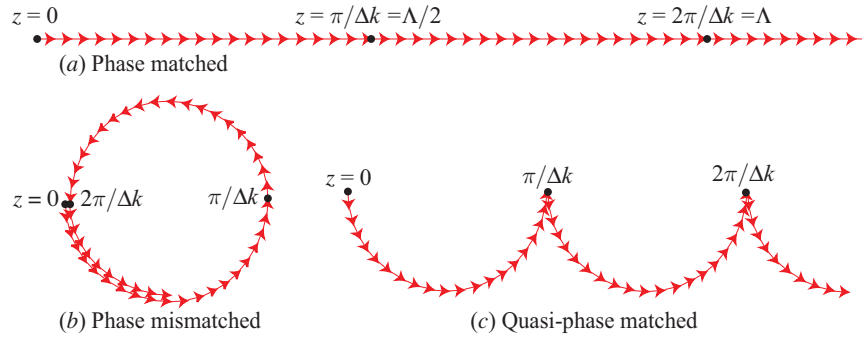


Figure 21.2-16 Phasors of the waves radiated by incremental elements at different positions z in the nonlinear medium. (a) In the phase-matched case ($\Delta k = 0$) the phasors are all aligned and maximum conversion efficiency is attained. (b) In the presence of a phase mismatch Δk , the phasors are misaligned and the efficiency is significantly reduced. (c) In the quasi-phase matched case, the misaligned phasors are periodically reversed by reversing the sign of the nonlinear coefficient at $\Lambda/2$ intervals. The conversion efficiency is partially restored.

for odd m , and zero, otherwise. If phase matching is accomplished via the m th harmonic, i.e., $\Lambda = mL_c$, then the parametric conversion efficiency is proportional to $d_m^2 = (2/\pi m)^2 d_o^2$. By contrast, a homogeneous medium with nonlinear coefficient d_o , the same length L , but with wavevector mismatch Δk , has a conversion efficiency $d_o^2 \text{sinc}^2(\Delta k L/2\pi) = d_o^2 \text{sinc}^2(L/L_c)$, which falls as $(d_o^2/\pi^2)(L_c/L)^2$ when $L \gg L_c$. Since $L_c = \Lambda/m$, the improvement of conversion efficiency is a factor of $4(L/\Lambda)^2$, i.e., is proportional to the square of the number of periods of the periodic structure. Clearly, the use of a periodic medium can offer a significant improvement in conversion efficiency.

The most challenging aspect of quasi-phase matching is the fabrication of the periodic nonlinear structure. A uniform nonlinear crystal may be altered periodically by reversing the principal axis direction in alternating layers, thus creating a d coefficient with alternating sign. This may be accomplished by lithographically exposing the crystal to a periodic electric field that reverses the direction of the crystal's permanent electric polarization, a technique called **poling**. This approach has been applied to ferroelectric crystals such as LiTaO_3 , KTP, and LiNbO_3 ; the latter has spawned a technology known as **periodically poled lithium niobate (PPLN)**. Semiconductor crystals such as GaAs also have been used for the same purpose.

21.3 THIRD-ORDER NONLINEAR OPTICS

In media possessing centrosymmetry, the second-order nonlinear term is absent since the polarization must reverse exactly when the electric field is reversed. The dominant nonlinearity is then of third order,

$$\mathcal{P}_{\text{NL}} = 4\chi^{(3)}\mathcal{E}^3 \quad (21.3-1)$$

(see Fig. 21.3-1) and the material is called a **Kerr medium**. Kerr media respond to optical fields by generating third harmonics and sums and differences of triplets of frequencies.

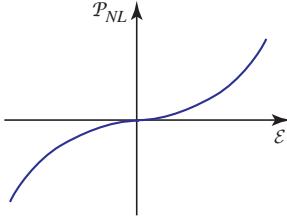


Figure 21.3-1 Third-order nonlinearity.

EXERCISE 21.3-1

Third-Order Nonlinear Optical Media Exhibit the Electro-Optic Effect, Kerr A monochromatic optical field $E(\omega)$ is incident on a third-order nonlinear medium in the presence of a steady electric field $E(0)$. The optical field is much smaller than the electric field, so that $|E(\omega)|^2 \ll |E(0)|^2$. Use (21.3-1) to show that the component of \mathcal{P}_{NL} of frequency ω is approximately given by $P_{\text{NL}}(\omega) \approx 12\chi^{(3)}E^2(0)E(\omega)$. Show that this component of the polarization is equivalent to a refractive-index change $\Delta n = -\frac{1}{2}s n^3 E^2(0)$, where

$$s = -\frac{12}{\epsilon_o n^4} \chi^{(3)}. \quad (21.3-2)$$

The proportionality between the refractive-index change and the squared electric field is the Kerr (quadratic) electro-optic effect described in Sec. 20.1A, where s is the Kerr coefficient.

A. Third-Harmonic Generation (THG) and Optical Kerr Effect**Third-Harmonic Generation (THG)**

In accordance with (21.3-1), the response of a third-order nonlinear medium to a monochromatic optical field $\mathcal{E}(t) = \text{Re}\{E(\omega) \exp(j\omega t)\}$ is a nonlinear polarization $\mathcal{P}_{\text{NL}}(t)$ containing a component at frequency ω and another at frequency 3ω ,

$$P_{\text{NL}}(\omega) = 3\chi^{(3)}|E(\omega)|^2 E(\omega) \quad (21.3-3a)$$

$$P_{\text{NL}}(3\omega) = \chi^{(3)}E^3(\omega). \quad (21.3-3b)$$

The presence of a component of polarization at the frequency 3ω indicates that third-harmonic light is generated. However, in most cases the energy conversion efficiency is low. Indeed, THG is often achieved via second-harmonic generation followed by sum-frequency generation of the fundamental and second-harmonic waves.

Optical Kerr Effect

The polarization component at frequency ω in (21.3-3a) corresponds to an incremental change of the susceptibility $\Delta\chi$ at frequency ω given by

$$\epsilon_o \Delta\chi = \frac{P_{\text{NL}}(\omega)}{E(\omega)} = 3\chi^{(3)}|E(\omega)|^2 = 6\chi^{(3)}\eta I, \quad (21.3-4)$$

where $I = |E(\omega)|^2/2\eta$ is the optical intensity of the initial wave. Since $n^2 = 1 + \chi$, we have $2n\Delta n = \Delta\chi$ so this is equivalent to an incremental refractive index $\Delta n = \Delta\chi/2n$:

$$\Delta n = \frac{3\eta}{\epsilon_o n} \chi^{(3)} I \equiv n_2 I, \quad (21.3-5)$$

where

$$n_2 = \frac{3\eta_o}{n^2\epsilon_o}\chi^{(3)}. \quad (21.3-6)$$

Optical Kerr Coefficient

Thus, the change in the refractive index is proportional to the optical intensity. The overall refractive index is therefore a linear function[†] of the optical intensity I ,

$$n(I) = n + n_2 I. \quad (21.3-7)$$

Optical Kerr Effect

This effect is known as the **optical Kerr effect** because of its similarity to the electro-optic Kerr effect discussed in Sec. 20.1A, for which Δn is proportional to the square of the steady electric field. The optical Kerr effect is a self-induced effect in which the phase velocity of the wave depends on the wave's own intensity. It is an example of **nonlinear refraction**.

The order of magnitude of the coefficient n_2 (in units of cm^2/W) is 10^{-16} to 10^{-14} in glasses, 10^{-14} to 10^{-7} in doped glasses, 10^{-10} to 10^{-8} in organic materials, and 10^{-10} to 10^{-2} in semiconductors. It is sensitive to the operating wavelength (see Sec. 21.7) and depends on the polarization.

B. Self-Phase Modulation (SPM), Self-Focusing, and Spatial Solitons

Self-Phase Modulation (SPM)

As a result of the optical Kerr effect, an optical wave traveling in a third-order nonlinear medium undergoes **self-phase modulation (SPM)**. The phase shift incurred by an optical beam of power P and cross-sectional area A , traveling a distance L in the medium, is $\varphi = -n(I)k_o L = 2\pi n(I)L/\lambda_o = -2\pi(n + n_2 P/A)L/\lambda_o$, so that it is altered by

$$\Delta\varphi = -2\pi n_2 \frac{L}{\lambda_o A} P, \quad (21.3-8)$$

which is proportional to the optical power P . Self-phase modulation is useful in applications in which light controls light.

To maximize the effect, L should be large and A small. These requirements are well served by the use of optical waveguides. The optical power at which $\Delta\varphi = -\pi$ is achieved is $P_\pi = \lambda_o A / 2Ln_2$. A doped-glass fiber of length $L = 1$ m, cross-sectional area $A = 10^{-2}$ mm^2 , and $n_2 = 10^{-10}$ cm^2/W , operating at $\lambda_o = 1$ μm , for example, switches the phase by a factor of π at an optical power $P_\pi = 0.5$ W. Materials with larger values of n_2 can be used in centimeter-long channel waveguides to achieve a phase shift of π at powers of a few mW.

Phase modulation may be converted into intensity modulation by employing one of the schemes used in conjunction with electro-optic modulators (see Sec. 20.1B): (1) using an interferometer (Mach-Zehnder, for example); (2) using the difference between the modulated phases of the two polarization components (birefringence) as a wave retarder placed between crossed polarizers; or (3) using an integrated-optic directional coupler (Sec. 8.5B). The result is an all-optical modulator in which a weak optical beam may be controlled by an intense optical beam. All-optical switches are discussed in Sec. 23.3C.

[†] Equation (21.3-7) is also written in the alternative form, $n(I) = n + n_2 |E|^2/2$, where n_2 differs from (21.3-6) by the factor η .

Self-Focusing

Another interesting effect associated with self-phase modulation is **self-focusing**. If an intense optical beam is transmitted through a thin sheet of nonlinear material exhibiting the optical Kerr effect, as illustrated in Fig. 21.3-2, the refractive-index change mimics the intensity pattern in the transverse plane. If the beam has its highest intensity at the center, for example, the maximum change of the refractive index is also at the center. The sheet then acts as a graded-index medium that imparts to the wave a nonuniform phase shift, thereby causing wavefront curvature. Under certain conditions the medium can act as a lens with a power-dependent focal length, as shown in Exercise 21.3-2. Kerr-lens focusing is useful for laser mode locking, as discussed in Sec. 15.4D.

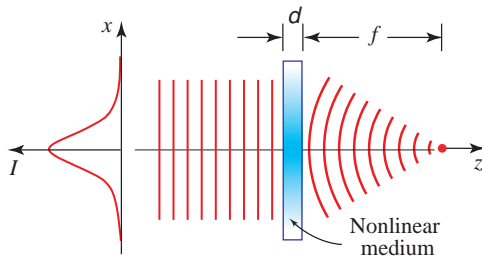


Figure 21.3-2 A third-order nonlinear medium acts as a lens whose focusing power depends on the intensity of the incident beam.

EXERCISE 21.3-2

Optical Kerr Lens. An optical beam traveling in the z direction is transmitted through a thin sheet of nonlinear optical material exhibiting the optical Kerr effect, $n(I) = n + n_2 I$. The sheet lies in the x - y plane and has a small thickness d so that its complex amplitude transmittance is $\exp(-jnk_0 d)$. The beam has an approximately planar wavefront and an intensity distribution $I \approx I_0[1 - (x^2 + y^2)/W^2]$ at points near the beam axis ($x, y \ll W$), where I_0 is the peak intensity and W is the beam width. Show that the medium acts as a thin lens with a focal length that is inversely proportional to I_0 . *Hint:* A lens of focal length f has a complex amplitude transmittance proportional to $\exp[jk_0(x^2 + y^2)/2f]$, as shown in (2.4-9); see also Exercise 2.4-6.

Spatial Solitons

When an intense optical beam travels through a substantial thickness of nonlinear homogeneous medium, instead of a thin sheet, the refractive index is altered nonuniformly so that the medium can act as a graded-index waveguide. Thus, the beam can create its own waveguide. If the intensity of the beam has the same spatial distribution in the transverse plane as one of the modes of the waveguide that the beam itself creates, the beam propagates self-consistently without changing its spatial distribution. Under these conditions, *diffraction* is compensated by *self-phase modulation*, and the beam is confined to its self-created waveguide. Such self-guided beams are called **spatial solitons**. Analogous behavior occurs in the time domain when group-velocity dispersion is compensated by self-phase modulation. As discussed in Sec. 22.5B, this leads to the formation of temporal solitons, which travel without changing shape.

The self-guiding of light in an optical Kerr medium is described mathematically by the Helmholtz equation, $\nabla^2 E + n^2(I)k_0^2 E = 0$, where $n(I) = n + n_2 I$, $k_0 = \omega/c_0$, and $I = |E|^2/2\eta$. This is a nonlinear differential equation in E , which is simplified by writing $E = A \exp(-jkz)$, where $k = nk_0$, and assuming that the envelope

$A = A(x, z)$ varies slowly in the z direction (in comparison with the wavelength $\lambda = 2\pi/k$) and does not vary in the y direction (see Sec. 2.2C). Using the approximation $(\partial^2/\partial z^2)[A \exp(-jkz)] \approx (-2jk \partial A/\partial z - k^2 A) \exp(-jkz)$, the Helmholtz equation becomes

$$\frac{\partial^2 A}{\partial x^2} - 2jk \frac{\partial A}{\partial z} + k_o^2 [n^2(I) - n^2] A = 0. \quad (21.3-9)$$

Since the nonlinear effect is small ($n_2 I \ll n$), we write

$$[n^2(I) - n^2] = [n(I) - n][n(I) + n] \approx [n_2 I][2n] = \frac{2n_2 n |A|^2}{2\eta} = \frac{n^2 n_2}{\eta_o} |A|^2, \quad (21.3-10)$$

so that (21.3-9) becomes

$$\frac{\partial^2 A}{\partial x^2} + \frac{n_2}{\eta_o} k^2 |A|^2 A = 2jk \frac{\partial A}{\partial z}. \quad (21.3-11)$$

Equation (21.3-11) is the nonlinear Schrödinger equation. One of its solutions is

$$A(x, z) = A_0 \operatorname{sech} \left(\frac{x}{W_0} \right) \exp \left(-j \frac{z}{4z_0} \right). \quad (21.3-12)$$

Spatial Soliton

where W_0 is a constant, $\operatorname{sech}(\cdot)$ is the hyperbolic-secant function, A_0 satisfies $n_2(A_0^2/2\eta_o) = 1/k^2 W_0^2$ and $z_0 = \frac{1}{2} k W_0^2 = \pi W_0^2/\lambda$ is the Rayleigh range [see (3.1-22)]. The intensity distribution

$$I(x, z) = \frac{|A(x, z)|^2}{2\eta} = \frac{A_0^2}{2\eta} \operatorname{sech}^2 \left(\frac{x}{W_0} \right) \quad (21.3-13)$$

is independent of z and has a width W_0 , as illustrated in Fig. 21.3-3. The distribution in (21.3-12) is the mode of a graded-index waveguide with a refractive index $n + n_2 I = n[1 + (1/k^2/W_0^2) \operatorname{sech}^2(x/W_0)]$, so that self-consistency is assured. Since $E = A \exp(-jkz)$, the wave travels with a propagation constant $k + 1/4z_0 = k(1 + \lambda^2/8\pi^2 W_0^2)$ and phase velocity $c/(1 + \lambda^2/8\pi^2 W_0^2)$. The velocity is smaller than c for localized beams (small W_0) but approaches c for large W_0 .

Raman Gain

The nonlinear coefficient $\chi^{(3)}$ is in general complex-valued, $\chi^{(3)} = \chi_R^{(3)} + j\chi_I^{(3)}$. The self-phase modulation in (21.3-8),

$$\Delta\varphi = 2\pi n_2 \frac{L}{\lambda_o A} P = \frac{6\pi\eta_o}{\epsilon_o} \frac{\chi^{(3)}}{n^2} \frac{L}{\lambda_o A} P, \quad (21.3-14)$$

is therefore also complex. Thus, the propagation phase factor $\exp(-j\varphi)$ is a combination of phase shift, $\Delta\varphi = (6\pi\eta_o/\epsilon_o)(\chi_R^{(3)}/n^2)(L/\lambda_o A)P$, and gain $\exp(\frac{1}{2}\gamma_R L)$, with

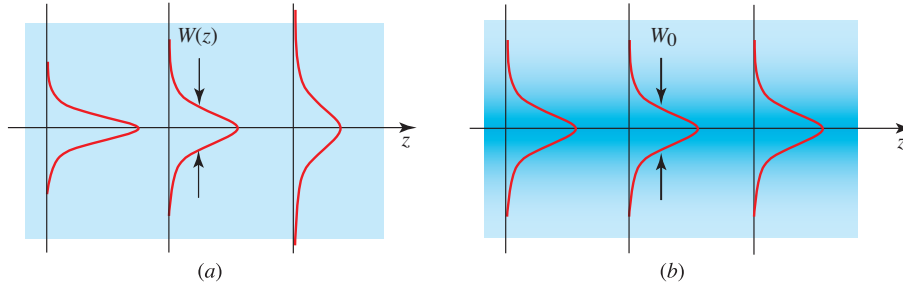


Figure 21.3-3 Comparison between (a) a Gaussian beam traveling in a linear medium, and (b) a spatial soliton (self-guided optical beam) traveling in a nonlinear medium.

a gain coefficient given by

$$\gamma_R = \frac{12\pi\eta_o \chi_I^{(3)}}{\epsilon_o n^2} \frac{1}{\lambda_o} \frac{P}{A}, \quad (21.3-15)$$

Raman Gain Coefficient

which is proportional to the optical power P . This effect, called **Raman gain**, has its origin in the coupling of light to the vibrational modes of a medium, which can act as an energy source. When this gain exceeds the loss, the medium can behave as an optical amplifier (see Sec. 14.3D). With proper feedback, the Raman amplifier becomes a Raman laser (see Sec. 15.3A). The phenomenological construct of a complex nonlinear coefficient $\chi^{(3)}$ is not unlike the complex susceptibility constructed to provide loss and gain in linear media (Sec. 5.5).

C. Cross-Phase Modulation (XPM)

We now consider the response of a third-order nonlinear medium to an optical field comprising two monochromatic waves of angular frequencies ω_1 and ω_2 , $\mathcal{E}(t) = \text{Re}\{E(\omega_1) \exp(j\omega_1 t)\} + \text{Re}\{E(\omega_2) \exp(j\omega_2 t)\}$. On substitution in (21.3-1), the component $P_{\text{NL}}(\omega_1)$ of the polarization density at frequency ω_1 turns out to be

$$P_{\text{NL}}(\omega_1) = \chi^{(3)} [3|E(\omega_1)|^2 + 6|E(\omega_2)|^2] E(\omega_1). \quad (21.3-16)$$

Assuming that the two waves have the same refractive index n , this relation may be cast in the form $P_{\text{NL}}(\omega_1) = 2\epsilon_o n \Delta n E(\omega_1)$, where

$$\Delta n = n_2(I_1 + 2I_2), \quad (21.3-17)$$

XPM

with $n_2 = 3\eta_o \chi^{(3)}/\epsilon_o n^2$. The quantities $I_1 = |E(\omega_1)|^2/2\eta$ and $I_2 = |E(\omega_2)|^2/2\eta$ are the intensities of waves 1 and 2, respectively. Therefore, wave 1 travels with an effective refractive index $n + \Delta n$ controlled by its own intensity as well as that of wave 2. Wave 2 encounters a similar effect, so that the waves are coupled.

Since the phase shift encountered by wave 1 is modulated by the intensity of wave 2, this phenomenon is known as **cross-phase modulation (XPM)**. It can result in the contamination of information between optical communication channels at neighboring frequencies, as in wavelength division multiplexing systems (WDM) (see Sec. 24.3C).

As we have seen in Sec. 21.2C, two-wave mixing is not possible in a second-order nonlinear medium (except in the degenerate case). Note, however, that two-wave mixing can occur in photorefractive media, as illustrated in Fig. 20.4-3.

EXERCISE 21.3-3

Optical Kerr Effect in the Presence of Three Waves. Three monochromatic waves with frequencies ω_1 , ω_2 , and ω_3 travel in a third-order nonlinear medium. Determine the complex amplitude of the component of $\mathcal{P}_{\text{NL}}(t)$ in (21.3-1) at frequency ω_1 . Show that this wave travels with a velocity $c_o/(n + \Delta n)$, where

$$\Delta n = n_2(I_1 + 2I_2 + 2I_3), \quad (21.3-18)$$

and $n_2 = 3\eta_o\chi^{(3)}/\epsilon_o n^2$, with $I_q = |E(\omega_q)|^2/2\eta$, $q = 1, 2, 3$.

D. Four-Wave Mixing (FWM)

We now examine the case of **four-wave mixing (FWM)** in a third-order nonlinear medium. We begin by determining the response of the medium to a superposition of three waves of angular frequencies ω_1 , ω_2 , and ω_3 , with field

$$\mathcal{E}(t) = \text{Re}\{E(\omega_1) \exp(j\omega_1 t)\} + \text{Re}\{E(\omega_2) \exp(j\omega_2 t)\} + \text{Re}\{E(\omega_3) \exp(j\omega_3 t)\}. \quad (21.3-19)$$

It is convenient to write $\mathcal{E}(t)$ as a sum of six terms

$$\mathcal{E}(t) = \sum_{q=\pm 1, \pm 2, \pm 3} \frac{1}{2} E(\omega_q) \exp(j\omega_q t), \quad (21.3-20)$$

where $\omega_{-q} = -\omega_q$ and $E(-\omega_q) = E^*(\omega_q)$. Substituting (21.3-20) into (21.3-1), we write \mathcal{P}_{NL} as a sum of $6^3 = 216$ terms,

$$\mathcal{P}_{\text{NL}}(t) = \frac{1}{8} \chi^{(3)} \sum_{q,r,l=\pm 1, \pm 2, \pm 3} E(\omega_q) E(\omega_r) E(\omega_l) \exp[j(\omega_q + \omega_r + \omega_l)t]. \quad (21.3-21)$$

Thus, \mathcal{P}_{NL} is the sum of harmonic components of frequencies $\omega_1, \dots, 3\omega_1, \dots, 2\omega_1 \pm \omega_2, \dots, \pm\omega_1 \pm \omega_2 \pm \omega_3$. The amplitude $P_{\text{NL}}(\omega_q + \omega_r + \omega_l)$ of the component of frequency $\omega_q + \omega_r + \omega_l$ can be determined by adding appropriate permutations of q , r , and l in (21.3-21). For example, $P_{\text{NL}}(\omega_1 + \omega_2 - \omega_3)$ involves six permutations,

$$P_{\text{NL}}(\omega_1 + \omega_2 - \omega_3) = 6\chi^{(3)} E(\omega_1) E(\omega_2) E^*(\omega_3), \quad (21.3-22)$$

Equation (21.3-22) indicates that four waves of frequencies ω_1 , ω_2 , ω_3 , and ω_4 are mixed by the medium if $\omega_4 = \omega_1 + \omega_2 - \omega_3$, or

$$\boxed{\omega_1 + \omega_2 = \omega_3 + \omega_4.} \quad (21.3-23)$$

Frequency-Matching Condition

This equation constitutes the frequency-matching condition for FWM.

Assuming that waves 1, 2, and 3 are plane waves of wavevectors \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{k}_3 , so that $E(\omega_q) \propto \exp(-j\mathbf{k}_q \cdot \mathbf{r})$, $q = 1, 2, 3$, then (21.3-22) gives

$$P_{\text{NL}}(\omega_4) \propto \exp(-j\mathbf{k}_1 \cdot \mathbf{r}) \exp(-j\mathbf{k}_2 \cdot \mathbf{r}) \exp(j\mathbf{k}_3 \cdot \mathbf{r}) = \exp[-j(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \cdot \mathbf{r}], \quad (21.3-24)$$

so that wave 4 is also a plane wave with wavevector $\mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$, from which

$$\boxed{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4.} \quad (21.3-25)$$

Phase-Matching Condition

Equation (21.3-25) is the phase-matching condition for FWM.

Several FWM processes occur simultaneously, all satisfying the frequency and phase matching conditions. As shown before, waves 1, 2, and 3 interact and generate wave 4, in accordance with (21.3-22). Similarly, waves 3, 4, and 1 interact and generate wave 2, in accordance with

$$P_{\text{NL}}(\omega_2) = 6\chi^{(3)}E(\omega_3)E(\omega_4)E^*(\omega_1), \quad (21.3-26)$$

and so on.

The FWM process may also be interpreted as an interaction between four photons. A photon of frequency ω_3 and another of frequency ω_4 are annihilated to create a photon of frequency ω_1 and another of frequency ω_2 , as illustrated in Fig. 21.3-4. Equations (21.3-23) and (21.3-25) represent conservation of energy and momentum, respectively.

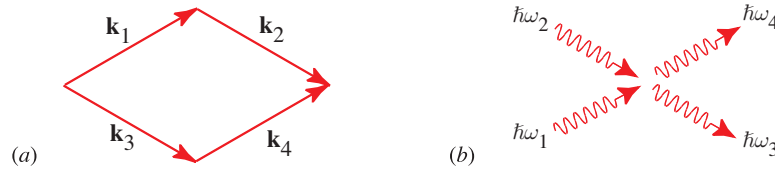


Figure 21.3-4 Four-wave mixing (FWM): (a) phase-matching condition; (b) interaction of four photons.

Three-Wave Mixing

In the partially degenerate case for which two of the four waves have the same frequency, $\omega_3 = \omega_4 \equiv \omega_0$, we have three waves with frequencies related by

$$\omega_1 + \omega_2 = 2\omega_0, \quad (21.3-27)$$

so that the frequencies ω_1 and ω_2 are symmetrically located with respect to the central frequency ω_0 , much like the sidebands of an amplitude modulated sine wave, or the Stokes and anti-Stokes frequencies in Raman scattering. The components of the nonlinear polarization density at ω_1 , ω_2 , and ω_3 include terms of the form

$$P_{\text{NL}}(\omega_1) = 3\chi^{(3)}E^2(\omega_3)E^*(\omega_2), \quad (21.3-28a)$$

$$P_{\text{NL}}(\omega_2) = 3\chi^{(3)}E^2(\omega_3)E^*(\omega_1), \quad (21.3-28b)$$

$$P_{\text{NL}}(\omega_3) = 6\chi^{(3)}E(\omega_1)E(\omega_2)E^*(\omega_3). \quad (21.3-28c)$$

These terms are responsible for three-wave mixing, i.e., radiation at the frequency of each wave generated by mixing of the other waves. These mixing processes may be used for optical frequency conversion (OFC), optical parametric amplification (OPA) and oscillation (OPO), and spontaneous parametric downconversion (SPDC), much like three-wave mixing in second-order nonlinear media; the waves at ω_1 , ω_2 , and ω_3 may be regarded as the signal, idler, and pump of the parametric process. Note, however, that this *three-wave mixing* process involves *four* photons. For example, the annihilation of two photons at ω_0 and the creation of two photons at ω_1 and ω_2 . An example of OPA in a $\chi^{(3)}$ medium, such as a silica-glass optical fiber, is illustrated in Fig. 21.3-5.

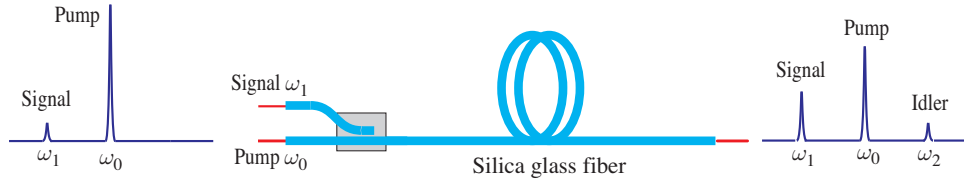


Figure 21.3-5 Three-wave, four-photon optical fiber parametric amplifier (OPA).

E. Optical Phase Conjugation (OPC)

The frequency-matching condition (21.3-23) is satisfied when all four waves are of the same frequency:

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega. \tag{21.3-29}$$

The process is then called **degenerate four-wave mixing**.

Assuming further that two of the waves (waves 3 and 4) are uniform plane waves traveling in opposite directions,

$$E_3(\mathbf{r}) = A_3 \exp(-j\mathbf{k}_3 \cdot \mathbf{r}), \quad E_4(\mathbf{r}) = A_4 \exp(-j\mathbf{k}_4 \cdot \mathbf{r}), \tag{21.3-30}$$

with

$$\mathbf{k}_4 = -\mathbf{k}_3, \tag{21.3-31}$$

and substituting (21.3-30) and (21.3-31) into (21.3-26), we see that the polarization density of wave 2 is $6\chi^{(3)}A_3A_4E_1^*(\mathbf{r})$. This term corresponds to a source emitting an optical wave (wave 2) of complex amplitude

$E_2(\mathbf{r}) \propto A_3A_4E_1^*(\mathbf{r}).$

$(21.3-32)$
 Phase Conjugation

Since A_3 and A_4 are constants, wave 2 is proportional to a conjugated version of wave 1. The device serves as a **phase conjugator**. Waves 3 and 4 are called the **pump** waves and waves 1 and 2 are called the **probe** and **conjugate** waves, respectively. As will be demonstrated shortly, the conjugate wave is identical to the probe wave except that it travels in the opposite direction. The phase conjugator is a special mirror that reflects the wave back onto itself without altering its wavefronts.

To understand the phase conjugation process consider two simple examples:

EXAMPLE 21.3-1. Conjugate of a Plane Wave. If wave 1 is a uniform plane wave, $E_1(\mathbf{r}) = A_1 \exp(-j\mathbf{k}_1 \cdot \mathbf{r})$, traveling in the direction \mathbf{k}_1 , then $E_2(\mathbf{r}) = A_1^* \exp(j\mathbf{k}_1 \cdot \mathbf{r})$ is a uniform plane wave traveling in the opposite direction $\mathbf{k}_2 = -\mathbf{k}_1$, as illustrated in Fig. 21.3-6(b). Thus, the phase-matching condition (21.3-25) is satisfied. The medium acts as a special “mirror” that reflects the incident plane wave back onto itself, no matter what the angle of incidence.

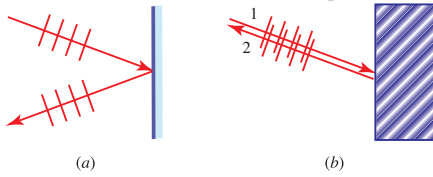


Figure 21.3-6 Reflection of a plane wave from (a) an ordinary mirror and (b) a phase conjugate mirror.

EXAMPLE 21.3-2. Conjugate of a Spherical Wave. If wave 1 is a spherical wave centered about the origin $\mathbf{r} = \mathbf{0}$, $E_1(\mathbf{r}) \propto (1/r) \exp(-jkr)$, then wave 2 has complex amplitude $E_2(\mathbf{r}) \propto (1/r) \exp(+jkr)$. This is a spherical wave traveling backward and converging toward the origin, as illustrated in Fig. 21.3-7(b).

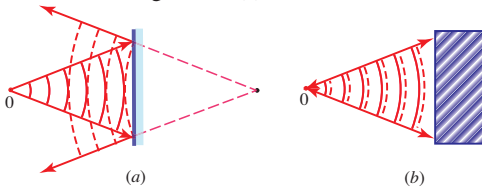


Figure 21.3-7 Reflection of a spherical wave from (a) an ordinary mirror and (b) a phase conjugate mirror.

Since an arbitrary probe wave may be regarded as a superposition of plane waves (see Chapter 4), each of which is reflected onto itself by the conjugator, the conjugate wave is identical to the incident wave everywhere, except for a reversed direction of propagation. The conjugate wave retraces the original wave by propagating backward, maintaining the same wavefronts.

Phase conjugation is analogous to time reversal. This may be understood by examining the field of the conjugate wave $\mathcal{E}_2(\mathbf{r}, t) = \text{Re}\{E_2(\mathbf{r}) \exp(j\omega t)\} \propto \text{Re}\{E_1^*(\mathbf{r}) \exp(j\omega t)\}$. Since the real part of a complex number equals the real part of its complex conjugate, $\mathcal{E}_2(\mathbf{r}, t) \propto \text{Re}\{E_1(\mathbf{r}) \exp(-j\omega t)\}$. Comparing this to the field of the probe wave $\mathcal{E}_1(\mathbf{r}, t) = \text{Re}\{E_1(\mathbf{r}) \exp(j\omega t)\}$, we readily see that one is obtained from the other by the transformation $t \rightarrow -t$, so that the conjugate wave appears as a time-reversed version of the probe wave.

The conjugate wave may carry more power than the probe wave. This can be seen by observing that the intensity of the conjugate wave (wave 2) is proportional to the product of the intensities of the pump waves 3 and 4 [see (21.3-32)]. When the powers of the pump waves are increased so that the conjugate wave (wave 2) carries more power than the probe wave (wave 1), the medium acts as an “amplifying mirror.” An example of an optical setup for demonstrating phase conjugation is shown in Fig. 21.3-8.

Degenerate Four-Wave Mixing as a Form of Real-Time Holography

The degenerate four-wave-mixing process is analogous to volume holography (see Sec. 4.5). Holography is a two-step process in which the interference pattern formed by the superposition of an object wave E_1 and a reference wave E_3 is recorded in a photographic emulsion. Another reference wave E_4 is subsequently transmitted through or reflected from the emulsion, creating the conjugate of the object wave $E_2 \propto E_4 E_3 E_1^*$, or its replica $E_2 \propto E_4 E_1 E_3^*$, depending on the geometry [see Fig 4.5-10(a) and (b)]. The nonlinear medium permits a real-time simultaneous holographic recording and reconstruction process. This process occurs in both the Kerr medium and the

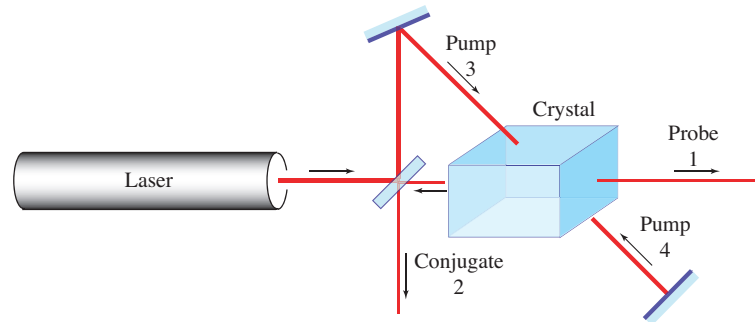


Figure 21.3-8 An optical system for degenerate four-wave mixing using a nonlinear crystal. The pump waves 3 and 4 and the probe wave 1 are obtained from a laser using a beamsplitter and two mirrors. The conjugate wave 2 is created within the crystal.

photorefractive medium (see Sec. 20.4).

When four waves are mixed in a nonlinear medium, each pair of waves interferes and creates a grating, from which a third wave is reflected to produce the fourth wave. The roles of reference and object are exchanged among the four waves, so that there are two types of gratings as illustrated in Fig. 21.3-9. Consider first the process illustrated in Fig. 21.3-9(a) [see also Fig. 4.5-10(a)]. Assume that the two reference waves (denoted as waves 3 and 4) are counterpropagating plane waves. The two steps of holography are:

1. The object wave 1 is added to the reference wave 3 and the intensity of their sum is recorded in the medium in the form of a volume grating (hologram).
2. The reconstruction reference wave 4 is Bragg reflected from the grating to create the conjugate wave (wave 2).

This grating is called the transmission grating.

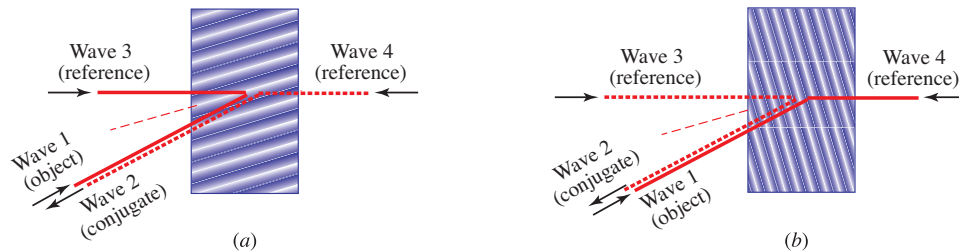


Figure 21.3-9 Four-wave mixing in a nonlinear medium. A reference and object wave interfere and create a grating from which the second reference wave reflects and produces a conjugate wave. There are two possibilities corresponding to (a) transmission and (b) reflection gratings.

The second possibility, illustrated in Fig. 21.3-9(b), is for the reference wave 4 to interfere with the object wave 1 and create a grating, called the reflection grating, from which the second reference wave 3 is reflected to create the conjugate wave 2. These two gratings can exist together but they usually have different efficiencies.

In summary, four-wave mixing can provide a means for real-time holography and phase conjugation, which have a number of applications in optical signal processing.

Use of Phase Conjugators in Wave Restoration

The ability to reflect a wave onto itself so that it retraces its path in the opposite direction suggests a number of useful applications, including the removal of wavefront

aberrations. The idea is based on the principle of reciprocity, illustrated in Fig. 21.3-10. Rays traveling through a linear optical medium from left to right follow the same path if they reverse and travel back in the opposite direction. The same principle applies to waves.

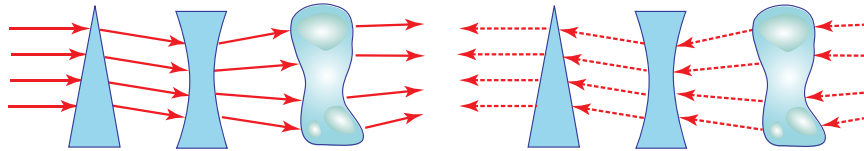


Figure 21.3-10 Optical reciprocity.

If the wavefront of an optical beam is distorted by an aberrating medium, the original wave can be restored by use of a conjugator that reflects the beam onto itself and transmits it once more through the same medium, as illustrated in Fig. 21.3-11.

One important application is in optical resonators (see Chapter 10). If the resonator contains an aberrating medium, replacing one of the mirrors with a conjugate mirror ensures that the distortion is removed in each round trip, so that the resonator modes have undistorted wavefronts transmitted through the ordinary mirror, as illustrated in Fig. 21.3-12.

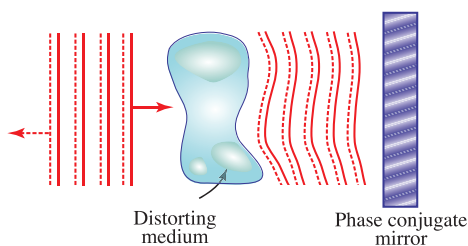


Figure 21.3-11 A phase conjugate mirror reflects a distorted wave onto itself, so that when it retraces its path, the distortion is compensated.

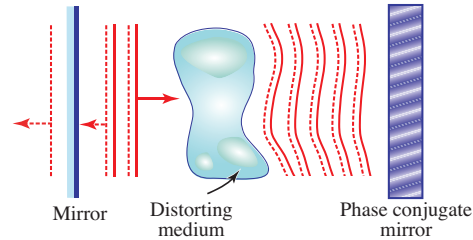


Figure 21.3-12 An optical resonator with an ordinary mirror and a phase conjugate mirror.

*21.4 SECOND-ORDER NONLINEAR OPTICS: COUPLED-WAVE THEORY

A quantitative analysis of the process of three-wave mixing in a second-order nonlinear optical medium is provided in this section using a coupled-wave theory. Unlike the treatment provided in Sec. 21.2, all three waves are treated on equal footing. To simplify the analysis, consideration of anisotropic and dispersive effects is deferred to Secs. 21.6 and 21.6, respectively.

Coupled-Wave Equations

In accordance with (21.1-6) and (21.1-7), wave propagation in a second-order nonlinear medium is governed by the basic wave equation

$$\nabla^2 \mathcal{E} - \frac{1}{c^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} = -\mathcal{S}, \tag{21.4-1}$$

where

$$\mathcal{S} = -\mu_o \frac{\partial^2 \mathcal{P}_{\text{NL}}}{\partial t^2} \quad (21.4-2)$$

is regarded as a radiation source, and

$$\mathcal{P}_{\text{NL}} = 2\mathbf{d}\mathcal{E}^2 \quad (21.4-3)$$

is the nonlinear component of the polarization density.

The field $\mathcal{E}(t)$ is a superposition of three waves of angular frequencies ω_1 , ω_2 , and ω_3 , with complex amplitudes E_1 , E_2 , and E_3 , respectively:

$$\mathcal{E}(t) = \sum_{q=1,2,3} \text{Re} \{ E_q \exp(j\omega_q t) \} = \sum_{q=1,2,3} \frac{1}{2} [E_q \exp(j\omega_q t) + E_q^* \exp(-j\omega_q t)]. \quad (21.4-4)$$

It is convenient to rewrite (21.4-4) in the compact form

$$\mathcal{E}(t) = \sum_{q=\pm 1, \pm 2, \pm 3} \frac{1}{2} E_q \exp(j\omega_q t), \quad (21.4-5)$$

where $\omega_{-q} = -\omega_q$ and $E_{-q} = E_q^*$. The corresponding polarization density obtained by substituting into (21.4-3) is a sum of $6 \times 6 = 36$ terms,

$$\mathcal{P}_{\text{NL}}(t) = 2\mathbf{d} \cdot \frac{1}{4} \sum_{q,r=\pm 1, \pm 2, \pm 3} E_q E_r \exp [j(\omega_q + \omega_r)t]. \quad (21.4-6)$$

Thus, the corresponding radiation source is

$$\mathcal{S} = \frac{1}{2} \mu_o \mathbf{d} \sum_{q,r=\pm 1, \pm 2, \pm 3} (\omega_q + \omega_r)^2 E_q E_r \exp [j(\omega_q + \omega_r)t], \quad (21.4-7)$$

which generates a sum of harmonic components whose frequencies are sums and differences of the original frequencies ω_1 , ω_2 , and ω_3 .

Substituting (21.4-5) and (21.4-7) into the wave equation (21.4-1) leads to a single differential equation with many terms, each of which is a harmonic function of some frequency. If the frequencies ω_1 , ω_2 , and ω_3 are distinct, we can separate this equation into three time-independent differential equations by equating terms on both sides of (21.4-1) at each of the frequencies ω_1 , ω_2 , and ω_3 , separately. The result is cast in the form of three Helmholtz equations with associated sources,

$$(\nabla^2 + k_1^2)E_1 = -S_1 \quad (21.4-8a)$$

$$(\nabla^2 + k_2^2)E_2 = -S_2 \quad (21.4-8b)$$

$$(\nabla^2 + k_3^2)E_3 = -S_3, \quad (21.4-8c)$$

where S_q is the amplitude of the component of \mathcal{S} with frequency ω_q and $k_q = n\omega_q/c_o$, $q = 1, 2, 3$. Each of the complex amplitudes of the three waves satisfies the Helmholtz equation with a source equal to the component of \mathcal{S} at its frequency. Under certain conditions, the source for one wave depends on the electric fields of the other two waves, so that the three waves are coupled.

In the absence of nonlinearity, $\mathbf{d} = 0$ so that the source term \mathcal{S} vanishes and each of the three waves satisfies the Helmholtz equation independently of the other two, as is expected in linear optics.

If the frequencies ω_1 , ω_2 , and ω_3 are not commensurate (one frequency is not the sum or difference of the other two, and one frequency is not twice another), then the source term \mathcal{S} does not contain any components of frequencies ω_1 , ω_2 , or ω_3 . The components S_1 , S_2 , and S_3 then vanish and the three waves do not interact.

For the three waves to be coupled by the medium, their frequencies must be commensurate. Assume, for example, that one frequency is the sum of the other two,

$$\omega_1 + \omega_2 = \omega_3. \quad (21.4-9)$$

The source \mathcal{S} then contains components at the frequencies ω_1 , ω_2 , and ω_3 . Examining the 36 terms of (21.4-7) yields

$$S_1 = 2\mu_o\omega_1^2 \mathbf{d} E_3 E_2^* \quad (21.4-10)$$

$$S_2 = 2\mu_o\omega_2^2 \mathbf{d} E_3 E_1^* \quad (21.4-11)$$

$$S_3 = 2\mu_o\omega_3^2 \mathbf{d} E_1 E_2. \quad (21.4-12)$$

The source for wave 1 is proportional to $E_3 E_2^*$ (since $\omega_1 = \omega_3 - \omega_2$), so that waves 2 and 3 together contribute to the growth of wave 1. Similarly, the source for wave 3 is proportional to $E_1 E_2$ (since $\omega_3 = \omega_1 + \omega_2$), so that waves 1 and 2 combine to amplify wave 3, and so on. The three waves are thus coupled or “mixed” by the medium in a process described by three coupled differential equations in E_1 , E_2 , and E_3 ,

$$(\nabla^2 + k_1^2)E_1 = -2\mu_o\omega_1^2 \mathbf{d} E_3 E_2^* \quad (21.4-13a)$$

$$(\nabla^2 + k_2^2)E_2 = -2\mu_o\omega_2^2 \mathbf{d} E_3 E_1^* \quad (21.4-13b)$$

$$(\nabla^2 + k_3^2)E_3 = -2\mu_o\omega_3^2 \mathbf{d} E_1 E_2. \quad (21.4-13c)$$

3-Wave-Mixing
Coupled Equations

EXERCISE 21.4-1

SHG as Degenerate Three-Wave Mixing. Equations (21.4-13) are valid only when the frequencies ω_1 , ω_2 , and ω_3 are distinct. Consider now the degenerate case for which $\omega_1 = \omega_2 = \omega$ and $\omega_3 = 2\omega$, so that there are two instead of three waves, with amplitudes E_1 and E_3 . This corresponds to second-harmonic generation (SHG). Show that these waves satisfy the Helmholtz equation with sources

$$S_1 = 2\mu_o\omega_1^2 \mathbf{d} E_3 E_1^* \quad (21.4-14)$$

$$S_3 = \mu_o\omega_3^2 \mathbf{d} E_1 E_1, \quad (21.4-15)$$

so that the coupled wave equations are

$$(\nabla^2 + k_1^2)E_1 = -2\mu_o\omega_1^2 \mathbf{d} E_3 E_1^*, \quad (21.4-16a)$$

$$(\nabla^2 + k_3^2)E_3 = -\mu_o\omega_3^2 \mathbf{d} E_1 E_1. \quad (21.4-16b)$$

SHG Coupled Equations

Note that these equations are not obtained from the three-wave-mixing equations (21.4-13) by substituting $E_1 = E_2$ [the factor of 2 is absent in (21.4-16b)].

Mixing of Three Collinear Uniform Plane Waves

Assume that the three waves are plane waves traveling in the z direction with complex amplitudes $E_q = A_q \exp(-jk_q z)$, complex envelopes A_q , and wavenumbers $k_q = \omega_q/c$, $q = 1, 2, 3$. It is convenient to normalize the complex envelopes by defining the variables $a_q = A_q/(2\eta\hbar\omega_q)^{1/2}$, where $\eta = \eta_o/n$ is the impedance of the medium, $\eta_o = (\mu_o/\epsilon_o)^{1/2}$ is the impedance of free space, and $\hbar\omega_q$ is the energy of a photon of angular frequency ω_q . Thus,

$$E_q = \sqrt{2\eta\hbar\omega_q} a_q \exp(-jk_q z), \quad q = 1, 2, 3, \quad (21.4-17)$$

and the intensities of the three waves are $I_q = |E_q|^2/2\eta = \hbar\omega_q |a_q|^2$. The photon flux densities (photons/s-m²) associated with these waves are

$$\phi_q = \frac{I_q}{\hbar\omega_q} = |a_q|^2. \quad (21.4-18)$$

The variable a_q therefore represents the complex envelope of wave q , scaled such that $|a_q|^2$ is the photon flux density. This scaling is convenient since the process of wave mixing must be governed by photon-number conservation (see Sec. 21.2C).

As a result of the interaction between the three waves, the complex envelopes a_q vary with z so that $a_q = a_q(z)$. If the interaction is weak, the $a_q(z)$ vary slowly with z , so that they can be assumed approximately constant within a distance of a wavelength. This makes it possible to use the slowly varying envelope approximation wherein $d^2 a_q/dz^2$ is neglected relative to $k_q da_q/dz = (2\pi/\lambda_q) da_q/dz$ and

$$(\nabla^2 + k_q^2)[a_q \exp(-jk_q z)] \approx -j2k_q \frac{da_q}{dz} \exp(-jk_q z) \quad (21.4-19)$$

(see Sec. 2.2C). With this approximation (21.4-13) reduce to simpler equations that are akin to the paraxial Helmholtz equations, in which the mismatch in phase is considered:

$$\frac{da_1}{dz} = -jg a_3 a_2^* \exp(-j\Delta k z) \quad (21.4-20a)$$

$$\frac{da_2}{dz} = -jg a_3 a_1^* \exp(-j\Delta k z) \quad (21.4-20b)$$

$$\frac{da_3}{dz} = -jg a_1 a_2 \exp(j\Delta k z) \quad (21.4-20c)$$

3-Wave-Mixing
Coupled Equations

where

$$g^2 = 2\hbar\omega_1\omega_2\omega_3\eta^3 d^2 \quad (21.4-21)$$

and

$$\Delta k = k_3 - k_2 - k_1 \quad (21.4-22)$$

represents the error in the phase-matching condition. The variations of a_1 , a_2 , and a_3 with z are therefore governed by three coupled first-order differential equations (21.4-20), which we proceed to solve under the different boundary conditions corresponding to various applications. It is useful, however, first to derive some invariants of the

wave-mixing process. These are functions of a_1 , a_2 , and a_3 that are independent of z . Invariants are useful since they can be used to reduce the number of independent variables. Exercises 21.4-3 and 21.4-2 develop invariants based on conservation of energy and conservation of photons.

EXERCISE 21.4-2

Photon-Number Conservation: The Manley–Rowe Relation. Using (21.4-20), show that

$$\frac{d}{dz}|a_1|^2 = \frac{d}{dz}|a_2|^2 = -\frac{d}{dz}|a_3|^2, \quad (21.4-23)$$

from which the Manley–Rowe relation (21.2-19), which was derived using photon-number conservation, follows. Equation (21.4-23) implies that $|a_1|^2 + |a_3|^2$ and $|a_2|^2 + |a_3|^2$ are also invariants of the wave-mixing process.

EXERCISE 21.4-3

Energy Conservation. Show that the sum of the intensities $I_q = \hbar\omega_q|a_q|^2$, $q = 1, 2, 3$, of the three waves governed by (21.4-20) is invariant to z , so that

$$\frac{d}{dz}(I_1 + I_2 + I_3) = 0. \quad (21.4-24)$$

A. Second-Harmonic Generation (SHG)

Second-harmonic generation (SHG) is a degenerate case of three-wave mixing in which

$$\omega_1 = \omega_2 = \omega \quad \text{and} \quad \omega_3 = 2\omega. \quad (21.4-25)$$

Two forms of interaction occur: two photons of frequency ω combine to form a photon of frequency 2ω (second harmonic), or one photon of frequency 2ω splits into two photons, each of frequency ω (degenerate parametric downconversion).

The interaction of the two waves is described by the paraxial Helmholtz equations with sources. Conservation of momentum requires that

$$2\mathbf{k}_1 = \mathbf{k}_3. \quad (21.4-26)$$

EXERCISE 21.4-4

Coupled-Wave Equations for SHG. Apply the slowly varying envelope approximation (21.4-19) to the Helmholtz equations (21.4-16), which describe two collinear waves in the degenerate case, to show that

$$\frac{da_1}{dz} = -jg a_3 a_1^* \exp(-j\Delta kz) \quad (21.4-27a)$$

$$\frac{da_3}{dz} = -j\frac{g}{2} a_1 a_1 \exp(j\Delta kz), \quad (21.4-27b)$$

where $\Delta k = k_3 - 2k_1$ and

$$g^2 = 4\hbar\omega^3\eta^3 d^2. \quad (21.4-28)$$

Assuming two collinear waves with perfect phase matching ($\Delta k = 0$), equations (21.4-27) reduce to

$$\frac{d\mathbf{a}_1}{dz} = -jg\mathbf{a}_3\mathbf{a}_1^* \quad (21.4-29a)$$

$$\frac{d\mathbf{a}_3}{dz} = -j\frac{g}{2}\mathbf{a}_1\mathbf{a}_1. \quad (21.4-29b)$$

SHG Coupled Equations

At the input to the device ($z = 0$) the amplitude of the second-harmonic wave is assumed to be zero, $\mathbf{a}_3(0) = 0$, and that of the fundamental wave, $\mathbf{a}_1(0)$, is assumed to be real. We seek a solution for which $\mathbf{a}_1(z)$ is real everywhere. Using the energy conservation relation $\mathbf{a}_1^2(z) + 2|\mathbf{a}_3(z)|^2 = \mathbf{a}_1^2(0)$, (21.4-29b) gives a differential equation in $\mathbf{a}_3(z)$,

$$d\mathbf{a}_3/dz = -j(g/2)[\mathbf{a}_1^2(0) - 2|\mathbf{a}_3(z)|^2], \quad (21.4-30)$$

whose solution may be substituted in (21.4-29a) to obtain the overall solution:

$$\mathbf{a}_1(z) = \mathbf{a}_1(0) \operatorname{sech}\left(\frac{1}{\sqrt{2}}g\mathbf{a}_1(0)z\right) \quad (21.4-31a)$$

$$\mathbf{a}_3(z) = -\frac{j}{\sqrt{2}}\mathbf{a}_1(0) \tanh\left(\frac{1}{\sqrt{2}}g\mathbf{a}_1(0)z\right). \quad (21.4-31b)$$

Consequently, the photon flux densities $\phi_1(z) = |\mathbf{a}_1(z)|^2$ and $\phi_3(z) = |\mathbf{a}_3(z)|^2$ evolve in accordance with

$$\phi_1(z) = \phi_1(0) \operatorname{sech}^2\frac{\gamma z}{2} \quad (21.4-32a)$$

$$\phi_3(z) = \frac{1}{2}\phi_1(0) \tanh^2\frac{\gamma z}{2}, \quad (21.4-32b)$$

where $\gamma/2 = g\mathbf{a}_1(0)/\sqrt{2}$, i.e.,

$$\gamma^2 = 2g^2\mathbf{a}_1^2(0) = 2g^2\phi_1(0) = 8d^2\eta^3\hbar\omega^3\phi_1(0) = 8d^2\eta^3\omega^2I_1(0). \quad (21.4-33)$$

Since $\operatorname{sech}^2(\cdot) + \tanh^2(\cdot) = 1$, $\phi_1(z) + 2\phi_3(z) = \phi_1(0)$ is constant, indicating that at each position z , photons of wave 1 are converted to half as many photons of wave 3. The fall of $\phi_1(z)$ and the rise of $\phi_3(z)$ with z are shown in Fig. 21.4-1(b).

Efficiency of SHG

The efficiency of second-harmonic generation for an interaction region of length L is

$$\eta_{\text{SHG}} = \frac{I_3(L)}{I_1(0)} = \frac{\hbar\omega_3\phi_3(L)}{\hbar\omega_1\phi_1(0)} = \frac{2\phi_3(L)}{\phi_1(0)} = \tanh^2\frac{\gamma L}{2}. \quad (21.4-34)$$

For large γL (long cell, large input intensity, or large nonlinear parameter), the efficiency approaches one. This signifies that all the input power (at frequency ω) has been transformed into power at frequency 2ω ; all input photons of frequency ω are converted into half as many photons of frequency 2ω .

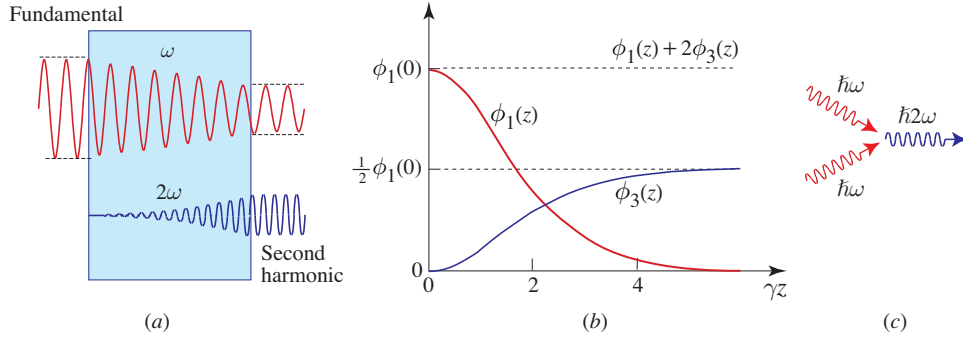


Figure 21.4-1 Second-harmonic generation. (a) A wave of frequency ω incident on a nonlinear crystal generates a wave of frequency 2ω . (b) As the photon flux density $\phi_1(z)$ of the fundamental wave decreases, the photon flux density $\phi_3(z)$ of the second-harmonic wave increases. Since photon numbers are conserved, the sum $\phi_1(z) + 2\phi_3(z) = \phi_1(0)$ is a constant. (c) Two photons of frequency ω combine to make one photon of frequency 2ω .

For small γL [small device length L , small nonlinear parameter d , or small input photon flux density $\phi_1(0)$], the argument of the tanh function is small and therefore the approximation $\tanh x \approx x$ may be used. The efficiency of second-harmonic generation is then

$$\eta_{\text{SHG}} = \frac{I_3(L)}{I_1(0)} \approx \frac{1}{4}\gamma^2 L^2 = \frac{1}{2}g^2 L^2 \phi_1(0) = 2d^2 \eta^3 \hbar \omega^3 L^2 \phi_1(0) = 2d^2 \eta^3 \omega^2 L^2 I_1(0), \quad (21.4-35)$$

so that

$$\eta_{\text{SHG}} = C^2 \frac{L^2}{A} P, \quad C^2 = 2\omega^2 \eta_o^3 \frac{d^2}{n^3}, \quad (21.4-36)$$

SHG Efficiency

where $P = I_1(0)A$ is the incident optical power at the fundamental frequency and A is the cross-sectional area. This reproduces (21.2-6) and shows that the constant C^2 is proportional to the material parameter d^2/n^3 , which is a figure of merit used for comparing different nonlinear materials.

EXAMPLE 21.4-1. Efficiency of SHG. For a material with $d^2/n^3 = 10^{-46} \text{ C/V}^2$ (see Table 21.6-3 for typical values of d) and a fundamental wave of wavelength $1 \mu\text{m}$, $C^2 = 38 \times 10^{-9} \text{ W}^{-1} = 0.038 (\text{MW})^{-1}$. In this case, the SHG efficiency is 10% if $PL^2/A = 2.63 \text{ MW}$. If the aspect ratio of the interaction volume is 1000, i.e., $L^2/A = 10^6$, the required power is 2.63 W. This may be realized using $L = 1 \text{ cm}$ and $A = 100 \mu\text{m}^2$, corresponding to a power density $P/A = 2.63 \times 10^6 \text{ W/cm}^2$. The SHG efficiency may be improved by using higher power density, longer interaction length, or material with greater d^2/n^3 coefficient.

Phase Mismatch in SHG

To study the effect of phase (or momentum) mismatch, the general equations (21.4-27) are used with $\Delta k \neq 0$. For simplicity, we limit ourselves to the weak-coupling case for which $\gamma L \ll 1$. In this case, the amplitude of the fundamental wave $a_1(z)$ varies

only slightly with z [see Fig. 21.4-1(a)], and may be assumed approximately constant. Substituting $a_1(z) \approx a_1(0)$ in (21.4-27b), and integrating, we obtain

$$a_3(L) = -j \frac{g}{2} a_1^2(0) \int_0^L \exp(j \Delta k z) dz = - \left(\frac{g}{2 \Delta k} \right) a_1^2(0) [\exp(j \Delta k L) - 1], \quad (21.4-37)$$

from which $\phi_3(L) = |a_3(L)|^2 = (g/\Delta k)^2 \phi_1^2(0) \sin^2(\Delta k L/2)$, where $a_1(0)$ is assumed to be real. The efficiency of second-harmonic generation is therefore

$$\eta_{\text{SHG}} = \frac{I_3(L)}{I_1(0)} = \frac{2\phi_3(L)}{\phi_1(0)} = C^2 \frac{L^2}{A} P \operatorname{sinc}^2(\Delta k L/2\pi), \quad (21.4-38)$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$.

The effect of phase mismatch is therefore to reduce the efficiency of second-harmonic generation by the factor $\operatorname{sinc}^2(\Delta k L/2\pi)$. This confirms the previous results displayed in Fig. 21.2-14. For a given mismatch Δk , the process of SHG is efficient for lengths smaller than the coherence length $L_c = 2\pi/|\Delta k|$.

B. Optical Frequency Conversion (OFC)

A frequency up-converter (Fig. 21.4-2) converts a wave of frequency ω_1 into a wave of higher frequency ω_3 by use of an auxiliary wave at frequency ω_2 , called the **pump**. A photon $\hbar\omega_2$ from the pump is added to a photon $\hbar\omega_1$ from the **signal** to form a photon $\hbar\omega_3$ of the **up-converted signal** at an up-converted frequency $\omega_3 = \omega_1 + \omega_2$.

The conversion process is governed by the three coupled equations (21.4-20). For simplicity, assume that the three waves are phase matched ($\Delta k = 0$) and that the pump is sufficiently strong so that its amplitude does not change appreciably within the interaction distance of interest; i.e., $a_2(z) \approx a_2(0)$ for all z between 0 and L . The three equations (21.4-20) then reduce to two,

$$\frac{da_1}{dz} = -j \frac{\gamma}{2} a_3 \quad (21.4-39a)$$

$$\frac{da_3}{dz} = -j \frac{\gamma}{2} a_1, \quad (21.4-39b)$$

where $\gamma = 2g a_2(0)$ and $a_2(0)$ is assumed real. These are simple differential equations with harmonic solutions

$$a_1(z) = a_1(0) \cos \frac{\gamma z}{2} \quad (21.4-40a)$$

$$a_3(z) = -j a_1(0) \sin \frac{\gamma z}{2}. \quad (21.4-40b)$$

The corresponding photon flux densities are

$$\phi_1(z) = \phi_1(0) \cos^2 \frac{\gamma z}{2} \quad (21.4-41a)$$

$$\phi_3(z) = \phi_1(0) \sin^2 \frac{\gamma z}{2}. \quad (21.4-41b)$$

The dependencies of the photon flux densities ϕ_1 and ϕ_3 on z are sketched in Fig. 21.4-2(b). Photons are exchanged periodically between the two waves. In the region between $z = 0$ and $z = \pi/\gamma$, the input ω_1 photons combine with the pump ω_2 photons and generate the up-converted ω_3 photons. Wave 1 is therefore attenuated, whereas wave 3 is amplified. In the region $z = \pi/\gamma$ to $z = 2\pi/\gamma$, the ω_3 photons are more abundant; they disintegrate into ω_1 and ω_2 photons, so that wave 3 is attenuated and wave 1 amplified. The process is repeated periodically as the waves travel through the medium.

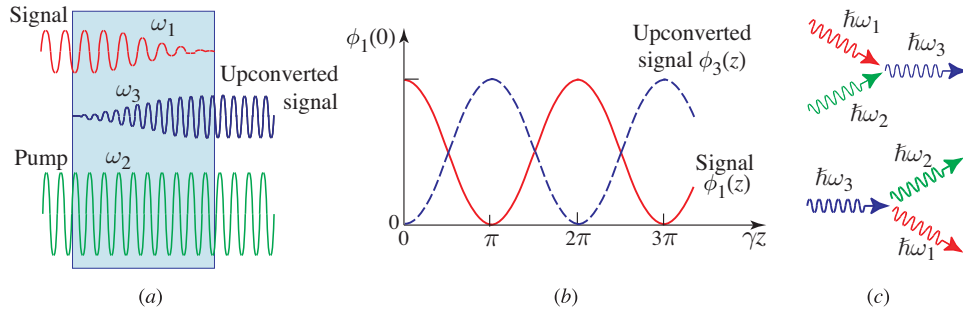


Figure 21.4-2 The frequency up-converter; (a) wave mixing; (b) evolution of the photon flux densities of the input ω_1 -wave and the up-converted ω_3 -wave. The pump ω_2 -wave is assumed constant; (c) photon interactions.

The efficiency of up-conversion for a device of length L is

$$\eta_{\text{OFC}} = \frac{I_3(L)}{I_1(0)} = \frac{\omega_3}{\omega_1} \sin^2 \frac{\gamma L}{2}. \quad (21.4-42)$$

For $\gamma L \ll 1$, and using (21.4-21), this is approximated by $I_3(L)/I_1(0) \approx (\omega_3/\omega_1) (\gamma L/2)^2 = (\omega_3/\omega_1) g^2 L^2 \phi_2(0) = 2\omega_3^2 L^2 d^2 \eta^3 I_2(0)$ from which

$$\eta_{\text{OFC}} = C^2 \frac{L^2}{A} P_2, \quad C^2 = 2\omega_3^2 \eta_o^3 \frac{d^2}{n^3}, \quad (21.4-43)$$

OFC Efficiency

where A is the cross-sectional area and $P_2 = I_2(0)A$ is the pump power. This expression is similar to (21.4-36) for the efficiency of second-harmonic generation.

EXERCISE 21.4-5

Infrared Up-Conversion. An up-converter uses a proustite crystal ($d = 1.5 \times 10^{-22} \text{ C/V}^2$, $n = 2.6$, $d^2/n^3 = 1.3 \times 10^{-45} \text{ C}^2/\text{V}^4$). The input wave is obtained from a CO_2 laser of wavelength $10.6 \mu\text{m}$, and the pump from a 1-W Nd^{3+} : YAG laser of wavelength $1.06 \mu\text{m}$ focused to a cross-sectional area 10^{-2} mm^2 (see Fig. 21.2-6). Determine the wavelength of the up-converted wave and the efficiency of up-conversion if the waves are collinear and the interaction length is 1 cm.

C. Optical Parametric Amplification (OPA) and Oscillation (OPO)

Optical Parametric Amplifier (OPA)

The OPA uses three-wave mixing in a nonlinear crystal to provide optical gain [Fig. 21.4-3(a)]. The process is governed by the same three coupled equations (21.4-20) with the waves identified as follows. Wave 1 is the **signal** to be amplified; it is incident on the crystal with a small intensity $I_1(0)$. Wave 3, the **pump**, is an intense wave that provides power to the amplifier. Wave 2, called the **idler**, is an auxiliary wave created by the interaction process.

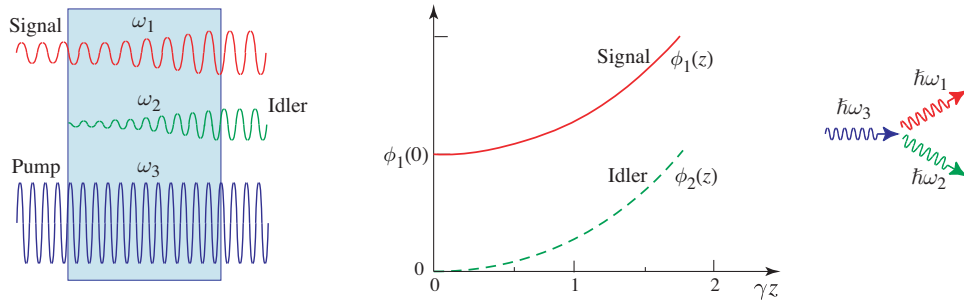


Figure 21.4-3 The optical parametric amplifier: (a) wave mixing; (b) photon flux densities of the signal and the idler (the pump photon-flux density is assumed constant); (c) photon mixing.

Assuming perfect phase matching ($\Delta k = 0$), and an undepleted pump, $a_3(z) \approx a_3(0)$, the coupled-wave equations (21.4-20) provide

$$\frac{da_1}{dz} = -j\frac{\gamma}{2}a_2^* \quad (21.4-44a)$$

$$\frac{da_2}{dz} = -j\frac{\gamma}{2}a_1^*, \quad (21.4-44b)$$

where $\gamma = 2g a_3(0)$. If $a_3(0)$ is real, γ is also real, and the differential equations have the solution

$$a_1(z) = a_1(0) \cosh \frac{\gamma z}{2} - j a_2^*(0) \sinh \frac{\gamma z}{2} \quad (21.4-45a)$$

$$a_2(z) = -j a_1^*(0) \sinh \frac{\gamma z}{2} + a_2(0) \cosh \frac{\gamma z}{2}. \quad (21.4-45b)$$

If $a_2(0) = 0$, i.e., the initial idler field is zero, then the corresponding photon flux densities are

$$\phi_1(z) = \phi_1(0) \cosh^2 \frac{\gamma z}{2} \quad (21.4-46a)$$

$$\phi_2(z) = \phi_1(0) \sinh^2 \frac{\gamma z}{2}. \quad (21.4-46b)$$

Both $\phi_1(z)$ and $\phi_2(z)$ grow monotonically with z , as illustrated in Fig. 21.4-3(b). This growth saturates when sufficient energy is drawn from the pump so that the assumption of an undepleted pump no longer holds.

The overall gain of an amplifier of length L is $G = \phi_1(L)/\phi_1(0) = \cosh^2(\gamma L/2)$. In the limit $\gamma L \gg 1$, $G = (e^{\gamma L/2} + e^{-\gamma L/2})^2/4 \approx e^{\gamma L}/4$, so that the gain increases exponentially with γL . The gain coefficient $\gamma = 2g\alpha_3(0) = 2d\sqrt{2\hbar\omega_1\omega_2\omega_3\eta^3}\alpha_3(0)$, from which

$$\gamma = 2C\sqrt{I_3(0)} = 2C\sqrt{P_3/A}, \quad C^2 = 2\omega_1\omega_2\eta_o^3\frac{d^2}{n^3}, \quad (21.4-47)$$

OPA Gain Coefficient

where $P_3 = I_3(0)A$ is the pump power and A is the cross-sectional area, and C^2 is a parameter similar to that describing SHG and OFC.

The interaction is tantamount to a pump photon $\hbar\omega_3$ splitting into a photon $\hbar\omega_1$ that amplifies the signal, and a photon $\hbar\omega_2$ that creates the idler [Fig. 21.4-3(c)].

EXERCISE 21.4-6

Gain of an OPA. An OPA amplifies light at $2.5 \mu\text{m}$ by using a 2-cm long KTP crystal pumped by a Nd:YAG laser of wavelength $1.064 \mu\text{m}$. Determine the wavelength of the idler wave and the C coefficient in (21.4-47). Determine appropriate laser power and beam cross-sectional area such that the total amplifier gain is 3 dB. Assume that $n = 1.75$ and $d = 2.3 \times 10^{-23} \text{ C/V}^2$ for KTP.

Optical Parametric Oscillator (OPO)

A parametric oscillator is constructed by providing feedback at either or both the signal and the idler frequencies of a parametric amplifier, as illustrated in Fig. 21.4-4. In the former case, the oscillator is called a **singly resonant oscillator (SRO)**; in the latter, it is called a **doubly resonant oscillator (DRO)**.

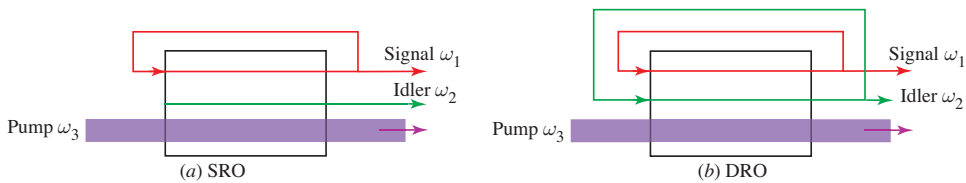


Figure 21.4-4 The parametric oscillator generates light at frequencies ω_1 and ω_2 . A pump of frequency $\omega_3 = \omega_1 + \omega_2$ serves as the source of energy. (a) Singly resonant oscillator (SRO). (b) Doubly resonant oscillator (DRO).

The oscillation frequencies ω_1 and ω_2 of the parametric oscillator are determined by the frequency- and phase-matching conditions, $\omega_1 + \omega_2 = \omega_3$ and $n_1\omega_1 + n_2\omega_2 = n_3\omega_3$, in the collinear case. The solution of these two equations yields ω_1 and ω_2 , as described in Sec. 21.2D. In addition, these frequencies must also coincide with the resonance frequencies of the resonator modes, much the same as for conventional lasers (see Sec. 15.1B). The system therefore tends to be over-constrained, particularly in the DRO case for which both the signal and idler frequencies must coincide with resonator modes.

Another condition for oscillation is that the gain of the amplifier must exceed the loss introduced by the mirrors for one round trip of propagation within the resonator. By equating the gain and the loss, expressions for the threshold amplifier gain and the corresponding threshold pump intensity may be determined, as shown below for the SRO and DRO configurations.

SRO. At the threshold of oscillation, the signal's amplified and doubly reflected amplitude $\alpha_1(L) r_1^2$ equals the initial amplitude $\alpha_1(0)$, where L is the length of the nonlinear medium and r_1 is the magnitude of the amplitude reflectance of a mirror (the two mirrors are assumed identical and the phase associated with a round trip is not included since it is a multiple of 2π). Using (21.4-45a), together with the boundary condition $\alpha_2(0) = 0$, we obtain $r_1^2 \cosh(\gamma L/2) = 1$, from which

$$\mathcal{R}_1^2 \cosh^2(\gamma L/2) = 1. \quad (21.4-48)$$

Here, $\mathcal{R}_1 = r_1^2$ is the mirror intensity reflectance at the signal frequency. Since \mathcal{R}_1 is typically slightly smaller than unity, $\cosh^2(\gamma L/2)$ is slightly greater than unity, i.e., $\gamma L/2 \ll 1$ and the approximation $\cosh^2(x) \approx 1 + x^2$ may be used. It follows that at threshold $(\gamma L/2)^2 \approx (1 - \mathcal{R}_1^2)/\mathcal{R}_1^2$. Using (21.4-47), we obtain the threshold intensity, from which the threshold power of the pump is obtained,

$$P_3|_{\text{threshold}}(0) \approx \frac{1}{C^2} \frac{A}{L^2} \frac{1 - \mathcal{R}_1^2}{\mathcal{R}_1^2}, \quad (21.4-49)$$

SRO Threshold Pump Power

where $C^2 = 2\omega_1\omega_2 \eta_0^3 d^2/n^3$ and A is the cross-sectional area. For example, if $L^2/A = 10^6$, $C^2 = 10^{-7} \text{ W}^{-1}$, and $\mathcal{R}_1 = 0.9$, then $P_3|_{\text{threshold}}(0) \approx 2.3 \text{ W}$.

DRO. At threshold, two conditions must be satisfied: $\alpha_1(L) r_1^2 = \alpha_1(0)$ and $\alpha_2(L) r_2^2 = \alpha_2(0)$, where r_1 and r_2 are the magnitudes of the amplitude reflectances of the mirrors at the signal and idler frequencies, respectively. Substituting for $\alpha_1(L)$ from (21.4-45a), and substituting for $\alpha_2(L)$ from (21.4-45b), and forming the conjugate, we obtain

$$(1 - \mathcal{R}_1) \cosh \frac{\gamma z}{2} \alpha_1(0) + j\mathcal{R}_1 \sinh \frac{\gamma z}{2} \alpha_2^*(0) = 0 \quad (21.4-50a)$$

$$-j\mathcal{R}_2 \sinh \frac{\gamma z}{2} \alpha_1(0) + (1 - \mathcal{R}_2) \cosh \frac{\gamma z}{2} \alpha_2^*(0) = 0, \quad (21.4-50b)$$

where $\mathcal{R}_1 = r_1^2$ and $\mathcal{R}_2 = r_2^2$ are the intensity reflectance of the mirrors at the signal and idler frequencies, respectively. Equating the values of the ratio $\alpha_1(0)/\alpha_2^*(0)$ obtained from (21.4-50a) and (21.4-50b), we obtain

$$\tanh^2(\gamma L/2) = (1 - \mathcal{R}_1)(1 - \mathcal{R}_2)/(\mathcal{R}_1\mathcal{R}_2). \quad (21.4-51)$$

Since the right-hand side of (21.4-51) is much smaller than unity, we can use the approximation $\tanh x \approx x$ and write $(\gamma L/2)^2 \approx (1 - \mathcal{R}_1)(1 - \mathcal{R}_2)/(\mathcal{R}_1\mathcal{R}_2)$, from which we obtain the threshold pump power:

$$P_3|_{\text{threshold}}(0) \approx \frac{1}{C^2} \frac{A}{L^2} \frac{(1 - \mathcal{R}_1)(1 - \mathcal{R}_2)}{\mathcal{R}_1\mathcal{R}_2}. \quad (21.4-52)$$

DRO Threshold Pump Power

The ratio of the threshold pump power for the DRO configuration, to that for the SRO configuration, as calculated from (21.4-49) and (21.4-52), is $(\mathcal{R}_1/\mathcal{R}_2)(1 - \mathcal{R}_2)/(1 + \mathcal{R}_1)$. Since $\mathcal{R}_1 \approx 1$ and $\mathcal{R}_2 \approx 1$, this is approximately equal to $(1 - \mathcal{R}_2)/2$, which is a small number. Thus, the threshold power for the DRO is substantially smaller than that for the SRO. Unfortunately, DROs are more sensitive to fluctuations of the resonator length because of the requirement that the oscillation frequencies of *both* the signal and the idler match resonator modes. DROs therefore often have poor stability and spiky spectra.

*21.5 THIRD-ORDER NONLINEAR OPTICS: COUPLED-WAVE THEORY

A. Four-Wave Mixing (FWM)

We now derive the coupled differential equations that describe FWM in a third-order nonlinear medium, using an approach similar to that employed in the three-wave mixing case in Sec. 21.4.

Coupled-Wave Equations

Four waves constituting a total field

$$\mathcal{E}(t) = \sum_{q=1,2,3,4} \text{Re}[E_q \exp(j\omega_q t)] = \sum_{q=\pm 1, \pm 2, \pm 3, \pm 4} \frac{1}{2} E_q \exp(j\omega_q t) \quad (21.5-1)$$

travel in a medium characterized by a nonlinear density

$$\mathcal{P}_{\text{NL}} = 4\chi^{(3)} \mathcal{E}^3. \quad (21.5-2)$$

The corresponding source of radiation, $\mathcal{S} = -\mu_o \partial^2 \mathcal{P}_{\text{NL}} / \partial t^2$, is therefore a sum of $8^3 = 512$ terms,

$$\mathcal{S} = \frac{1}{2} \mu_o \chi^{(3)} \sum_{q,p,r=\pm 1, \pm 2, \pm 3, \pm 4} (\omega_q + \omega_p + \omega_r)^2 E_q E_p E_r \exp[j(\omega_q + \omega_p + \omega_r)t]. \quad (21.5-3)$$

Substituting (21.5-1) and (21.5-3) into the wave equation (21.4-1) and equating terms at each of the four frequencies ω_1 , ω_2 , ω_3 , and ω_4 , we obtain four Helmholtz equations with associated sources,

$$(\nabla^2 + k_q^2) E_q = -S_q, \quad q = 1, 2, 3, 4, \quad (21.5-4)$$

where S_q is the amplitude of the component of \mathcal{S} at frequency ω_q .

For the four waves to be coupled, their frequencies must be commensurate. Consider, for example, the case for which the sum of two frequencies equals the sum of the other two frequencies,

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad (21.5-5)$$

and assume that these frequencies are distinct. Three waves can then combine and create a source at the fourth frequency. Using (21.5-5), terms in (21.5-3) at each of the