Title of the Paper
Date of Examination : Order Statistics

Let $x_1, x_2, \ldots x_n$ be n independent and identically distributed variates, each with cumulative distn function $F(x)$. If these variables are arranged in ascending order of magnitude and then written as $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$, we call $x_{(r)}$ as the r^{th} order statistic, $r = 1, 2, \ldots n$.

The $x_{(r)}$'s because of the inequality relations among them are necessarily dependent.

Note:- If we write these ordered values as

$$Y_1 \le Y_2 \le \ldots \le Y_n \text{ then}$$

$$Y_1 = x_{(1)} = \min(x_1, x_2, \ldots x_n)$$

$$\vdots$$

$$Y_r = x_{(r)} = r^{th} \text{ smallest of } x_1, x_2, \ldots x_n.$$

$$\vdots$$

$$Y_n = x_{(n)} = \max(x_1, x_2, \ldots x_n)$$

Cumulative Distribution Function of a single order statistic:-

Let $F_r(x)$, $r = 1, 2, \ldots n$ denote the cdf of the r^{th} order statistic $x_{(r)}$

The the cdf of the largest order statistic $x_{(n)}$ is given by

$$F_n(x) = P(x_{(n)} \le x) = P(x_i \le x;\ i = 1, 2 \ldots n)$$

$$= P(x_1 \le x \cap x_2 \le x \cap \ldots \cap x_n \le x)$$

$$= P(x_1 \le x) \cdot P(x_2 \le x) \ldots P(x_n \le x)$$

$\because x_i$'s are indep.

$$= (F(x))^n.$$

Since $x_1, x_2, \ldots x_n$ are identically distd.

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The cdf of the smallest order statistic $x_{(1)}$ is given by,

$$F_1(x) = P(X_{(1)} \le x)$$
$$= 1 - P(X_{(1)} > x)$$
$$= 1 - P(X_i > x,\ i = 1, 2 \dots n)$$
$$= 1 - \prod_{i=1}^{n} P(X_i > x)$$
$$= 1 - \prod_{i=1}^{n} \left(1 - P(X_i \le x)\right)$$
$$= 1 - \left(1 - F(x)\right)^n.$$

Since $X_1, X_2, \dots X_n$ are i.i.d. r.v's.

In general, the cdf of the r^{th} order statistic is given by,

$$F_r(x) = P(X_{(r)} \le x)$$
$$= P(\text{At least } r \text{ of the } x_i\text{'s are} \le x)$$
$$= \sum_{j=r}^{n} P(\text{Exactly } j \text{ of the } n,\ x_i\text{'s are} \le x)$$
$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} \left(F(x)\right)^j \left(1 - F(x)\right)^{n-j} \longrightarrow (*)$$

by using Binomial probability model.

Note:

1) The above expression (*) can also be written as

$$F_r(x) = I_{F(x)}(r, n-r+1)$$

where $I_p(a, b) = \frac{1}{\beta(a,b)} \int_0^p t^{a-1} (1-t)^{b-1}\, dt$

is the incomplete Beta function.

2) Taking $r = 1$, in (*) we get the cdf of the smallest-order statistic $F_1(x)$ as follows.

$$F_1(x) = \sum_{j=1}^{n} \binom{n}{j} \left[F(x)\right]^j \left[1 - F(x)\right]^{n-j}$$

$$F_1(x) = \sum_{j=0}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j} - \binom{n}{j}(F(x))^j (1-F(x))^{n-j} \Big/_{j=0}$$ ③

$$= 1 - (1-F(x))^n$$

i.e. $F_1(x) = 1-(1-F(x))^n$. $\Rightarrow f_1(x) = n(1-F(x))^{n-1} f(x)$.

Substitute $r=n$, in ④, we get the cdf of largest order statistic $F_n(x)$ as follows,

$$F_n(x) = \sum_{j=n}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j}$$

$$= \binom{n}{n} (F(x))^n (1-F(x))^{n-n}$$

$\therefore$ $F_n(x) = (F(x))^n$. $\rightarrow f_n(x) = n(F(x))^{n-1} f(x)$.

PDF of a single order statistic :-

Let us assume that x_i's are i.i.d continuous r.v's with pdf $f(x) = \frac{d}{dx} F(x)$. If $f_r(x)$ denotes the pdf of $x_{(r)}$ then

$$f_r(x) = \frac{d}{dx}(F_r(x))$$

$$= \frac{d}{dx}\left(I_{F(x)}(r, n-r+1)\right)$$

$$= \frac{d}{dx}\left(\frac{1}{\beta(r, n-r+1)} \int_0^{F(x)} t^{r-1}(1-t)^{n-r} dt\right) \rightarrow ①$$

Let us write,

$$g(t) = \int t^{r-1}(1-t)^{n-r} dt$$

$$\rightarrow g'(t) = t^{r-1}(1-t)^{n-r} \longrightarrow ②$$

Consider, $\int_0^{F(x)} t^{r-1}(1-t)^{n-r} dt$

$$= g(t)\Big|_0^{F(x)}$$

$$= g(F(x)) - g(0) \longrightarrow ③$$

where $g(0)$ is a constant.

Again Consider

$$\frac{d}{dx}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}\,dt$$

Using (3), the above expression can be rewritten as

$$\frac{d}{dx}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}\,dt = \cancel{g'(F(x))}$$

$$= \frac{d}{dx}\left[g(F(x)) - g(0)\right]$$

$$= g'(F(x))\cdot f(x). \longrightarrow (4).$$

$$= (F(x))^{r-1}(1-F(x))^{n-r} f(x) \quad (\because \text{Using } (2)).$$

Therefore, the pdf of the r^{th} order statistic $f_r(x)$ can be written as

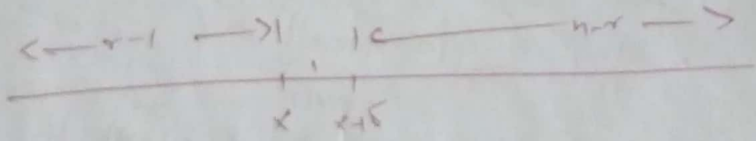
$$f_r(x) = \frac{1}{\beta(r, n-r+1)} (F(x))^{r-1}(1-F(x))^{n-r} f(x) \longrightarrow (5).$$

Alternate proof:

By definition of pdf, we have,

$$f_r(x) = \lim_{\delta x\to 0} \frac{P(x< X_{(r)} \le x+\delta x)}{\delta x} \longrightarrow (6).$$

The event $E: x < X_{(r)} \le x+\delta x$ can be materialise as follows



$x_i \le x$ for $(r-1)$ of the x_i's.

$x < x_i \le x+\delta x$ for one x_i and

$x_i \ge x+\delta x$ for the remaining $(n-r)$ of the x_i's.

Hence, by the multinomial probability law, we have

$$P(x< X_{(r)} \le x+\delta x) = \frac{n!}{(r-1)!\,1!\,(n-r)!}\, p_1^{r-1}\, p_2^{1}\, p_3^{n-r} \longrightarrow (7)$$

where $p_1 = P(X_i \le x) = F(x)$

$p_2 = P(x < X_i \le x+\delta x) = F(x+\delta x) - F(x)$

$\Rightarrow$ $p_3 = P(X_i \ge x+\delta x) = 1 - P(X_i \le x+\delta x) = 1 - F(x+\delta x)$

Substituting the above in (7) & (6), we get:

$$f_r(x) = \lim_{\delta x \to 0} \frac{n!}{(r-1)!\,1!\,(n-r)!} \cdot \left(F(x)\right)^{r-1} \cdot \left(\frac{F(x+\delta x) - F(x)}{\delta x}\right) \left(1-F(x+\delta x)\right)^{n-r}$$

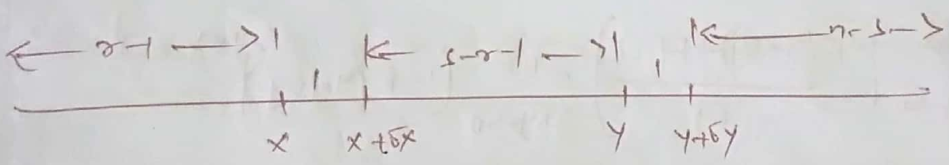
$$\therefore f_r(x) = \frac{1}{\beta(r, n-r+1)} \left(F(x)\right)^{r-1} f(x) \cdot \left(1-F(x)\right)^{n-r} \rightarrow (8).$$

Joint pdf of two order statistic :-

Let us denote the joint pdf of $X_{(r)}$ and $X_{(s)}$, where $1 \le r \le s \le n$ by $f_{rs}(x, y)$. Then

$$f_{rs}(x, y) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{P\left(x \le X_{(r)} \le x+\delta x \cap y \le X_{(s)} \le y+\delta y\right)}{\delta x \cdot \delta y} \rightarrow (**)$$

The Event $E = x \le X_{(r)} \le x+\delta x \cap y \le X_{(s)} \le y+\delta y$ can be materialise as follows,



$x_i \le x$ for $r-1$ of the x_i's,

$x < x_i \le x+\delta x$ for one x_i.

$x+\delta x < x_i \le y$ for $(s-r-1)$ of x_i's,

$y < x_i \le y+\delta y$ for one x_i. and

$x_i > y+\delta y$ for $(n-s)$ of the x_i's.

Hence, by using the multinomial probability law, we can get

$$P(E) = P\left(x \le X_{(r)} \le x+\delta x \cap y \le X_{(s)} \le y+\delta y\right)$$

$$P(E) = \frac{n!}{(r-1)!\,1!\,(s-r-1)!\,1!\,(n-s)!} \; p_1^{r-1} \, p_2^{1} \, p_3^{s-r-1} \, p_4^{1} \, p_5^{n-s} \rightarrow (***)$$

where $p_1 = P(X_i \le x) = F(x)$

$$p_2 = P(x < X_i \le x+\delta x) = F(x+\delta x) - F(x)$$

$$p_3 = P(x+\delta x \le X_i \le y) = F(y) - F(x+\delta x)$$

$$p_4 = P(y \le X_i \le y+\delta y) = F(y+\delta y) - F(y).$$

$$p_5 = P(X_i \ge y+\delta y) = 1 - P(X_i \le y+\delta y) = 1 - F(y+\delta y).$$

Now, substituting the above in (**) and (***), we get:

$$f_{rs}(x,y) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{P(E)}{\delta x \cdot \delta y}.$$

$$= \frac{n!}{(r-1)!\,(n-s)!\,(s-r-1)!} \left(F(x)\right)^{r-1} \lim_{\delta x \to 0} \left(\frac{F(x+\delta x) - F(x)}{\delta x}\right)$$

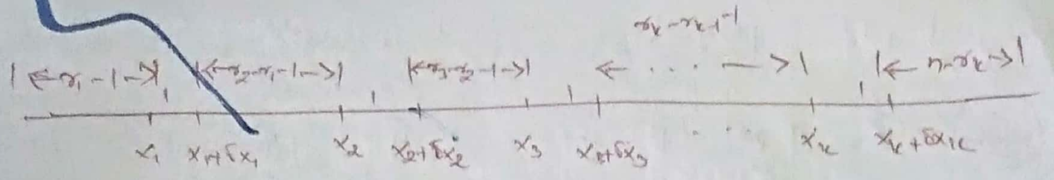
$$\lim_{\delta y \to 0} \left(\frac{F(y+\delta y) - F(y)}{\delta y}\right) \cdot \lim_{\delta y \to 0} \left(1 - F(y+\delta y)\right)^{n-s}$$

$$\lim_{\delta x \to 0} \left(F(y) - F(x+\delta x)\right)^{s-r-1}$$

$$\therefore f_{rs}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} \left(F(x)\right)^{r-1} f(x) \left(F(y) - F(x)\right)^{s-r-1} f(y) \left(1 - F(y)\right)^{n-s}$$

Joint pdf of k-order statistics:-

The joint pdf of k order statistics $x_{(r_1)}, x_{(r_2)} \ldots x_{(r_k)}$ where $1 \le r_1 < r_2 < \ldots < r_k \le n$ and $1 \le k \le n$ is by $x_1 \le x_2 \le \ldots \le x_k$ given by [on using the following configuration and multinomial prob. law)



$$f_{r_1,r_2,\dots,r_k}(x_1,x_2,\dots,x_k) = \frac{n!}{(r_1-1)!\,(r_2-r_1-1)!\cdots(r_k-r_{k-1}-1)!\,(n-r_k)!}$$

$$\cdot\,(F(x_1))^{r_1-1} f(x_1)\,(F(x_2)-F(x_1))^{r_2-r_1-1} f(x_2)\,(F(x_3)-F(x_2))^{r_3-r_2-1} f(x_3)$$

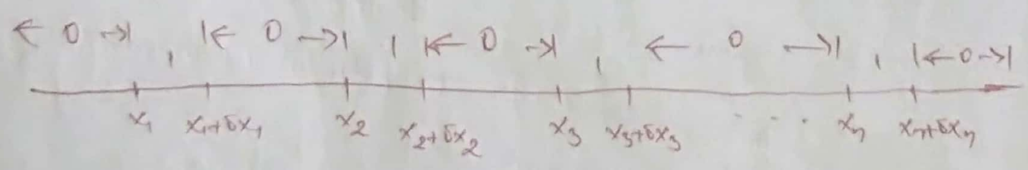
$$\cdots (1-F(x_k))^{n-r_k}\, f(x_k). \longrightarrow (a)$$

Joint pdf of all n-order statistics :-

In particular the joint pdf of all the n order statistics is obtained on taking $k=n$ in (a). This implies that $r_i = i$, for $i=1,2,\dots,n$. Hence the joint pdf of $X_{(1)}, X_{(2)},\dots, X_{(n)}$ is given by,

$$f_{1,2,\dots,n}(x_1,x_2,\dots,x_n) = n!\, f(x_1)\, f(x_2)\cdots f(x_n). \longrightarrow (b)$$

We can easily obtain (b) by using the following configuration



Distribution of Range :-

Let us obtain the pdf of the statistic $W_{rs} = X_{(s)} - X_{(r)}$; $r<s$.

We have, the joint pdf of $X_{(r)}$ and $X_{(s)}$ is,

$$f_{rs}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}(F(x))^{r-1} f(x)\,(F(y)-F(x))^{s-r-1} f(y)\,(1-F(y))^{n-s}.$$

Transform $(X_{(r)}, X_{(s)})$ to the new variables W_{rs} & $X_{(r)}$ as

$$w_{rs} = y-x \quad \text{and} \quad x = x.$$

$$\Rightarrow \quad y = x + w_{rs} \quad \text{and} \quad x = x.$$

$$\therefore\ J = \frac{\partial(x,y)}{\partial(x,w_{rs})} = \begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} = 1 \Rightarrow |J| = 1.$$

The joint pdf $f_{rs}(x,y)$ transforms to the joint pdf of $X_{(r)}$ and W_{rs} as (8)

$$g(x, w_{rs}) = C_{rs}\,(F(x))^{r-1} f(x)\,(F(x+w_{rs}) - F(x))^{s-r-1} f(x+w_{rs})\,(1-F(x+w_{rs}))^{n-s}$$

where $C_{rs} = \dfrac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}$

Integrating the above w.r.to x from $-\infty$ to ∞, we get the pdf of w_{rs} as

$$g(w_{rs}) = C_{rs}\int_{-\infty}^{\infty} (F(x))^{r-1} f(x)\,(F(x+w_{rs}) - F(x))^{s-r-1} f(x+w_{rs})\,(1-F(x+w_{rs}))^{n-s}\,dx \rightarrow (*)$$

<u>Distn of Range:-</u> $(w = X_{(n)} - X_{(1)})$

Taking $r=1$ and $s=n$ in $(*)$, we obtain the pdf of the range $w = X_{(n)} - X_{(1)}$ as

$$g(w) = n(n-1)\int_{-\infty}^{\infty} f(x)\,(F(x+w) - F(x))^{n-2} f(x+w)\,dx. \rightarrow (1).$$

The cdf of w is given by,

$$G(w) = P(W \le w) = \int_0^{w} g(u)\,du$$

$$G(w) = \int_0^{w} n(n-1)\cdot\int_{-\infty}^{\infty} f(x)\,(F(x+u) - F(x))^{n-2} f(x+u)\,dx\,du.$$

$$= n\int_{-\infty}^{\infty} f(x)\int_0^{w} (n-1)\, f(x+u)\,(F(x+u) - F(x))^{n-2}\,du\,dx.$$

$$\therefore\; G(w) = n\int_{-\infty}^{\infty} f(x)\cdot(F(x+w) - F(x))^{n-1}\,dx. \rightarrow (2).$$

let $F(x+u) - F(x) = v$.

$F(x+u) = v + F(x)$

$x+u = F^{-1}(v+F(x))$

$u = F^{-1}(v+F(x)) - x$

uv-ful.

~~$uv^{1} + vu^{1}$~~

~~$f(x+u)(n-2)\,(F(x+u) - F(x))^{n-3}$~~

$$(n-1)\left(f(x+u)\cdot\frac{(F(x+u)-F(x))^{n-1}}{n-1} - \int \frac{(F(x+u)-F(x))^{n-1}}{n-1}\, f'(x+u)\,du\right.$$

Eg:- Show that for a random sample of size 2 from $N(0, \sigma^2)$ population, $E[X_{(1)}] = -\sigma/\sqrt{\pi}$.

Sol:- For $n=2$, the pdf $f_1(x)$ of $X_{(1)}$ is given by,

$$f_1(x) = \frac{1}{\beta(1,2)} \left(1 - F(x)\right) f(x)$$

$$= 2\left(1 - F(x)\right) f(x) \; ; \; -\infty < x < \infty$$

where $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ $\quad (\because X \sim N(0, \sigma^2))$

$$\therefore E[X_{(1)}] = \int_{-\infty}^{\infty} x \cdot f_1(x)\, dx$$

$$= 2 \int_{-\infty}^{\infty} x \underbrace{\left[1 - F(x)\right]}_{u} \underbrace{f(x)\, dx}_{v} \quad \longrightarrow (1)$$

$$\cancel{= 2\left\{ \left[1-F(x)\right] F(x) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} F(x) f(x)\, dx \right\}}$$

we have, $\log f(x) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{x^2}{2\sigma^2}$

differentiating above w.r.to. x we get,

$$\frac{f'(x)}{f(x)} = -\frac{x}{\sigma^2} \Rightarrow -\sigma^2 f'(x) = x f(x).$$

Integrate w.r.to. x, we get,

$$-\sigma^2 \int f'(x)\, dx = \int x f(x)\, dx.$$

$$\therefore \int x f(x)\, dx = -\sigma^2 \int \frac{d}{dx} f(x)\, dx$$

$$= -\sigma^2 f(x). \quad \longrightarrow (2)$$

Integrating (1) by parts and using (2), we get

$$E(X_{(1)}) = 2\left(\left[1 - F(x)\right]\left(-\sigma^2 f(x)\right)\Big|_{-\infty}^{\infty}\right) - 2\int_{-\infty}^{\infty} \left(-\sigma^2 f(x)\right)\left(-f(x)\right) dx.$$

Eg: Show that in odd samples of size n from $U[0,1]$ popln

the mean and variance of the distn of median are $\frac{1}{2}$ and $\frac{1}{4(n+2)}$ resp.

Sol: we have, $f(x) = 1 \; ; \; 0 \le x \le 1$

$$F(x) = P(X \le x) = \int_0^x f(u)\,du$$

$$= \int_0^x 1 . du = x.$$

Let $n = 2m+1$ (odd), where m is a positive integer ≥ 1.

Then median observation is $X_{(m+1)}$. Taking $r = (m+1)$ in pdf of $X_{(r)}$,

$$f_r(x) = \frac{1}{\beta(r, n-r+1)} \cdot [F(x)]^{r-1} [1-F(x)]^{n-r} f(x).$$

The pdf of median $X_{(m+1)}$ is given by,

$$f_{m+1}(x) = \frac{1}{\beta(m+1, m+1)} \cdot x^m (1-x)^m \cdot 1.$$

$$\therefore E(X_{(m+1)}) = \frac{1}{\beta(m+1, m+1)} \int_0^1 x . x^m (1-x)^m dx$$

$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

$$= \frac{1}{\beta(m+1, m+1)} \cdot \int_0^1 x^{m+1} (1-x)^m dx.$$

$B(\alpha,\beta) = \frac{\Gamma\alpha \, \Gamma\beta}{\Gamma(\alpha+\beta)}$

$$= \frac{1}{\beta(m+1, m+1)} \cdot \int_0^1 x^{(m+2)-1} (1-x)^{(m+1)-1} dx$$

$$= \frac{\beta(m+2, m+1)}{\beta(m+1, m+1)} = \frac{\Gamma(m+2)\,\cancel{\Gamma(m+1)}}{\Gamma(2m+3)} \cdot \frac{\Gamma(2m+2)}{\Gamma(m+1)\,\cancel{\Gamma(m+1)}}$$

$$= \frac{(m+1) . \cancel{\Gamma(m+1)} . \cancel{\Gamma(2m+2)}}{(2m+2)\,\cancel{\Gamma(2m+2)}\,\cancel{\Gamma(m+1)}} = \frac{m+1}{2m+2} = \frac{1}{2}$$

$$E(X^2_{(m+1)}) = \int_0^1 x^2 f_{m+1}(x)\,dx = \int_0^1 x^2 x^m (1-x)^m dx = \int_0^1 x^{m+2} (1-x)^m dx.$$

$$= \frac{\beta(m+3, m+1)}{\beta(m+1, m+1)} = \frac{m+2}{2(2m+3)}$$

$$\therefore \; Var(X_{(m+1)}) = E(X_{(m+1)}^2) - [E(X_{(m+1)})]^2$$

$$= \frac{m+2}{2(2m+3)} - \frac{1}{4} = \frac{1}{4(2m+3)}$$

$$= \frac{1}{4(n+2)}$$

Eg:- Let X_1, X_2, and X_3 be a random sample of size 3 from

$$f(x) = 2x \;;\; 0<x<1.$$

obtain the density of Y_1, the first order statistics. Also find $P(Y_1 > \frac{1}{2})$.

Sol:- $n=3, \quad f(x) = 2x \;;\; 0<x<1.$

$$F(x) = P(X \le x) = \int_0^x f(u)\,du$$

$$= \int_0^x 2u\,du = 2\left(\frac{u^2}{2}\right)_0^x = x^2.$$

The pdf of 1st order statistic Y_1 is,

$$f_1(y) = n\,[1-F(x)]^{n-1} f(x).$$

$$= 3\,(1-x^2)^2\, 2x = 6x(1-x^2)^2.$$

$$G_1(y) = P(Y_1 \le y) = 1 - P(Y_1 \ge y)$$

$$\cancel{= 1 - 6x(1-x^2)^2}$$

$$\cancel{= 1 - 6y(1-y^2)^2}$$

$\cancel{\therefore P(Y_1 \le \frac{1}{2})}$

$6u(1-u^2)^2$

$6u(1+u^4-2u^2)$

$6[u+u^5-2u^3]$

$$P(Y_1 \le y) = \int_0^y f(u)\,du = \int_0^y 6u(1-u^2)^2\,du$$

$$= 6\int_0^y (u+u^5-2u^3)\,du = 6\left(\frac{u^2}{2} + \frac{u^6}{6} - 2\frac{u^4}{4}\right)_0^y$$

$$= 6\left(\frac{y^2}{2} + \frac{y^6}{6} - \frac{y^4}{2}\right) = 3y^2 + y^6 - 3y^4$$

$$P(Y_1 \ge y) = 1 - (3y^2 + y^6 - 3y^4) \Rightarrow P(Y_1 \ge \tfrac{1}{2}) = 1 - \left[3\cdot\tfrac{1}{4} + \tfrac{1}{2^6} - 3\cdot\tfrac{1}{2^4}\right]$$

Eg:- Let $Y_1 < Y_2 < Y_3$ denote the order statistics of a random sample of size 3 from a distn having pdf. (12)

$$f(x) = \begin{cases} 1 & ; 0 < x < 1 \\ 0 & ; \text{otherwise.} \end{cases}$$

Find the joint density of Range and Midrange.

Sol:-

$n = 3.$ $f(x) = 1.$

$F(x) = x.$

The joint density of range and midrange is,

$$g(x, w_{rs}) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} \left[F(x)\right]^{r-1} f(x) \left(F(x+w_{rs}) - F(x)\right)^{s-r-1} f(x+w_{rs}) \left[1-F(x+w_{rs})\right]^{n-s}$$

$$= n(n-1)\, f(x). \left(\underset{F(y)-F(x).}{F(x+w) - F(x)}\right)^{n-2} \underset{f(y).}{f(x+w)}$$

$$= n(n-1) \cdot 1 \cdot (x+w-x)^{n-2} f(x+w)$$

$$= 6\,w\,f(x+w).$$ $$= n(n-1)\,(y-x)^{n-2}.$$

Eg:- a) Find the pdf of $X_{(r)}$ in a random sample of size n from the exponential distn

$$f(x) = \alpha e^{-\alpha x} \quad ; \alpha > 0, \; x \geq 0.$$

b) Show that $X_{(r)}$ and $W_{rs} = X_{(s)} - X_{(r)}$, $r < s$, are independently distd.

c) What is the distn of $W_r = X_{(r+1)} - X_{(r)}$?

Sol:-

Here $F(x) = P(X \leq x) = \int_0^x \alpha \cdot e^{-\alpha u} du = 1 - e^{-\alpha x}$

The pdf of $X_{(r)}$ is given by

$$f_r(x) = \frac{1}{\beta(r, n-r+1)} \left(F(x)\right)^{r-1} \left(1-F(x)\right)^{n-r} f(x).$$

$$= \frac{1}{\beta(r, n-r+1)} \cdot \left(1 - e^{-\alpha x}\right)^{r-1} e^{-\alpha x(n-r)} \, \alpha \cdot e^{-\alpha x}$$

$$f_r(x) = \frac{1}{\beta(r, n-r+1)} \cdot \alpha \cdot e^{-\alpha x(n-r+1)} \cdot (1-e^{-\alpha x})^{r-1} \quad ; \ x > 0.$$

b) The joint pdf of $X_{(r)}$ and $W_{rs} = X_{(s)} - X_{(r)}$ is given by

$$g(x, w_{rs}) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} [F(x)]^{r-1} f(x) [F(x+w_{rs}) - F(x)]^{s-r-1} f(x+w_{rs}) [1-F(x+w_{rs})]^{n-s}$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{(n-r)!}{(s-r-1)!\,(n-s)!} (1-e^{-\alpha x})^{r-1} \alpha e^{-\alpha x}$$

$$\left(e^{-\alpha x} - e^{-\alpha(x+w_{rs})}\right)^{s-r-1} \alpha e^{-\alpha(x+w_{rs})} \left(e^{-\alpha(x+w_{rs})}\right)^{n-s}$$

$$= \left[\frac{1}{\beta(r, n-r+1)} \cdot \alpha \cdot e^{-\alpha x(\cancel{s}-r-\cancel{1}+\cancel{1}+1+n-\cancel{s})} (1-e^{-\alpha x})^{r-1}\right]$$

$$\left[\frac{1}{\beta(s-r, n-s+1)} \cdot \alpha \cdot e^{-\alpha w_{rs}(n-s+1)} (1-e^{-\alpha w_{rs}})^{s-r-1}\right] \rightarrow (*).$$

$\therefore$ $X_{(r)}$ and W_{rs} are independently distributed.

c) Taking $s = r+1$ in $(*)$, the pdf of $W_1 = X_{(r+1)} - X_{(r)}$ becomes,

$$g(w_1) = \frac{1}{\beta(1, n-r)} \cdot \alpha\, e^{-\alpha(n-r) \cdot w_1}$$

$$= (n-r) \cdot \alpha \cdot e^{-\alpha(n-r) w_1} \quad ; \ w_1 \geq 0.$$

which shows that w_1 has an exponential distn with parameter $(n-r) \cdot \alpha$.

$$E(X_{(1)}) = -2\sigma^2 \int_{-\infty}^{\infty} (f(x))^2 dx = -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-x^2/\sigma^2} dx.$$

$$= -\frac{1}{\pi} \cdot \frac{\sqrt{\pi}}{(1/\sigma)} = -\sigma/\sqrt{\pi}. \quad \left(\because \int_{-\infty}^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{a}\right)$$

Eg:- Let $x_1, x_2, \dots x_n$ be a random sample with common pdf

$$f(x) = \begin{cases} 1 & ; 0<x<1 \\ 0 & ; \text{otherwise.} \end{cases}$$

i) Find the pdf, mean and variance of $X_{(1)}$.

ii) Find the pdf, mean and variance of $X_{(n)}$.

iii) Find $Corr(X_{(1)}, X_{(n)})$.

Sol:-

i) Given $f(x) = 1 \quad ; 0<x<1$

$$F(x) = x \quad ; x \geq 0.$$

i) $f_1(x) = n\left(1-F(x)\right)^{n-1} f(x)$

$$= n(1-x)^{n-1}.$$

$$E(X_{(1)}) = \int_0^1 x\, f_1(x)\,dx$$

$$= \int_0^1 n \cdot x^1 (1-x)^{n-1}\,dx$$

$$= n \cdot B(2,n) = n \cdot \frac{\Gamma 2\, \Gamma n}{\Gamma(n+2)} = \frac{n\, \Gamma n}{(n+1)\, \Gamma n} = \frac{n}{n+1}$$

$$E(X_{(1)}^2) = \int_0^1 x^2 f_1(x)\,dx.$$

$$= \int_0^1 n \cdot x^2 (1-x)^{n-1}\,dx$$

$(3-1)(3-2)\Gamma(3-3)\Gamma(3-3)$

$\frac{n \cdot 2 \cdot (n-1)!}{(n+2)!}$ $(n-2)!$

$(n+3-1)(n+3-2)(n+3-3)\Gamma(n+3-3)$

$(n+2)(n+1) \cdot n$.

$$= n\, B(3,n) = n \frac{\Gamma 3\, \Gamma n}{\Gamma(n+3)} = \frac{n\, 2 \cdot \Gamma n}{(n+2)(n+1)\, \Gamma n}$$

$$= \frac{2n}{(n+1)(n+2)}$$

$$Var(X_{(1)}) = E(X_{(1)}^2) - \left(E(X_{(1)})\right)^2$$

$$= \frac{2n}{(n+1)(n+2)} - \frac{n^2}{(n+1)^2} = \frac{n}{n+1}\left(\frac{2}{n+2} - \frac{n}{n+1}\right)$$

$$= \frac{n}{n+1}\left[\frac{2n+2-n^2-2n}{(n+2)(n+1)}\right]$$

$\frac{2}{(n+1)(n+2)} \quad \frac{n^2}{(n+1)^2}$

$\frac{n}{n+1}\left(\frac{2}{n+2} - \frac{n^2}{n+1}\right)$

$\frac{n}{n+1}\left(\frac{2n+2-n^3-2n^2}{(n+2)(n+1)}\right)$

$$Var(X_{(1)}) = \frac{n}{(n+2)(n+1)^2}.$$

ii) $f_n(x) = nx^{n-1}$; $0 \le x \le 1$.

$$E(X_{(n)}) = \frac{n}{n+1}$$

$$Var(X_{(n)}) = \frac{n}{(n+2)(n+1)^2}.$$

iii) $$Corr(X_{(1)}, X_{(n)}) = \frac{Cov(X_{(1)}, X_{(n)})}{\sqrt{Var(X_{(1)}) \cdot Var(X_{(n)})}}.$$

$$E(X_{(1)} \cdot X_{(n)}) = \int_0^y \int_0^1 xy\, f_{1n}(x, y)\, dx\, dy$$

$$= n(n-1) \int_0^y \int_0^1 xy (y-x)^{n-2}\, dx\, dy$$

$\because f_{1n}(x, y) = n(n-1) f(x) \left(F(y) - F(x)\right)^{n-2} f(y)$; $0 \le x \le 1$.

$$\therefore E(X_{(1)} \cdot X_{(n)}) = n(n-1) \int_0^1 \int_0^1 x \cdot y^{n-1} \left(1 - \frac{x}{y}\right)^{n-2} dx \cdot dy.$$

$$= n(n-1) \int_0^1 \int_0^1 y^{n+1}\, t \cdot (1-t)^{n-2}\, dt\, dy \quad \left(\because \frac{x}{y} = t.\right)$$

$$= \frac{1}{n+2}.$$

$$Cov(X_{(1)}, X_{(n)}) = \frac{1}{(n+1)^2 (n+2)}.$$ and

$$Corr(X_{(1)}, X_{(n)}) = \frac{1}{n}.$$