Th:- Cochran's Theorem:- Imp

Stat:- Let $x_1, x_2, \dots x_n$ denote a random sample from $N(0, \sigma^2)$.

Let the sum of squares of these x_i's can be written as,

$$\sum_{i=1}^{n} x_i^2 = Q_1 + Q_2 + \dots + Q_k$$

where Q_i is the quadratic form in $x_1, x_2, \dots x_n$ with matrix A_j which has rank r_j, $j = 1, 2, \dots k$. Then the r.v. $Q_1, Q_2, \dots Q_k$ are mutually stochastically independent and $\frac{Q_j}{\sigma^2} \sim \chi^2(r_j)$, $j = 1, 2, \dots k$ iff $\sum_{j=1}^{k} r_j = n$.

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(92)

proof :- Assume that $\sum_{j=1}^{k} r_j = n$ and

$$\sum_{i=1}^{n} x_i^2 = Q_1 + Q_2 + \cdots + Q_k$$

prove that Q_j, $j = 1, 2, \ldots k$ are independent and

$$\frac{Q_j}{\sigma^2} \sim \chi^2(r_j).$$

$Q = x' A x$

$x^2 = x x'$
$= x I x'$

Consider, $$\sum_{i=1}^{n} x_i^2 = \sum_{j=1}^{k} Q_j$$

$$x' I x = x' A_1 x + x' A_2 x + \cdots + x' A_k x$$

$$x' I x = x' (A_1 + A_2 + \cdots + A_k) x$$

$$\therefore \quad I = A_1 + A_2 + \cdots + A_k$$

Let $B_i = I - A_i \quad \Rightarrow \quad I = A_i + B_i$

In otherwords, B_i is the sum of the matrices $A_1, A_2 \ldots A_k$ excluding A_i.

Let R_i denote the rank of the matrix $B_i \Rightarrow \text{Rank}(B_i) = R_i$

we have, $$\text{Rank}(B_i) \leq \Sigma \text{Rank}(A_1, A_2, \ldots A_k \text{ excluding } A_i)$$

$$R_i \leq (r_1 + r_2 + \cdots + r_k) - r_i$$

$$R_i \leq \sum_{j=1}^{k} r_j - r_i$$

$$\therefore \quad R_i \leq n - r_i \quad \longrightarrow (1)$$

However, $I = A_i + B_i$

$$\text{Rank}(I) \leq \text{Rank}(A_i) + \text{Rank}(B_i)$$

$$\text{Rank}(A_1 + A_2 + \cdots + A_k) \leq r_i + R_i$$

$$\Sigma r_j \leq r_i + R_i$$

$$n \leq r_i + R_i$$

$$\therefore \quad n - r_i \leq R_i \quad \longrightarrow (2)$$

From (1) and (2), we have

$$R_i = n - r_i \quad \longrightarrow (3)$$

The characteristic eqn of matrix B_i is denoted as $|B_i - \lambda I| = 0$ $\longrightarrow (4)$

Since $B_i = I - A_i$

Eqn (4) can be written as

$$|I - A_i - \lambda I| = 0$$

$$|-(A_i - I + \lambda I)| = 0$$

$$(-1)^n |A_i - I + \lambda I| = 0$$

$$\therefore |A_i - (1-\lambda) I| = 0 \quad \longrightarrow (5)$$

Now, by comparing (4)th and (5)th, we conclude that the characteristic roots of A_i equal to $1 - B_i$.

We recall that the rank of the matrix is equal to the no. of non-zero characteristic roots.

$\therefore$ The matrix B_i has exactly $(n - r_i)$ characteristic roots.

Explanation: we know that $\text{Rank}(A_i) = r_i$

$\text{Rank}(B_i) = R_i = n - r_i$ (∵ using (3))

But $\text{Rank}(B_i) = R_i = n - r_i$ non-zero characteristic roots.

Since order of matrix B_i is n, it has n charn roots. $\longrightarrow (6)$

So $n - (n - r_i) = r_i$ only zero charn roots. Therefore A_i has r_i non-zero charn roots (has $\text{Rank}(A_i) = r_i$). But, there are r_i non-zero charn roots $\sigma = 1$ using (6).

In other words, the roots of A_i are either zero or one.

$\Rightarrow$ A_i is an idempotent matrix. (102)

$Q_i = x' A_i x$

i.e. $A_i^2 = A_i$

$x_1, x_2, \ldots x_n \sim N$

Thus $\dfrac{Q_i}{\sigma^2} \sim \chi^2(r_i)$, $i = 1, 2, \ldots k$.

$\dfrac{\Sigma x_i^2}{\sigma^2} \sim \chi^2(n)$

Converse part:-

Assume that $\sum_{i=1}^{n} x_i^2 = Q_1 + Q_2 + \cdots + Q_k$

where $Q_1, Q_2, \ldots Q_k$ are mutually stochastically independent and

$$\frac{Q_j}{\sigma^2} \sim \chi^2(r_j), \quad j = 1, 2, \ldots k.$$

$\Sigma x_i^2 = \Sigma Q_j$

To prove $\sum_{j=1}^{k} r_j = n$.

$\dfrac{\Sigma x_i^2}{\sigma^2} = \dfrac{\Sigma Q_j}{\sigma^2}$

It is given that $\dfrac{Q_j}{\sigma^2} \sim \chi^2(r_j)$

$\Sigma \dfrac{Q_j}{\sigma^2} \sim \chi^2(n)$

$\dfrac{\Sigma x_i^2}{\sigma^2} = \Sigma \dfrac{Q_j}{\sigma^2}$

$$\therefore \sum_{j=1}^{k} \frac{Q_j}{\sigma^2} \sim \chi^2\left(\sum_{j=1}^{k} r_j\right) \quad (\because \text{Reproductive property of } \chi^2.) \longrightarrow (7)$$

$$\text{But } \sum_{j=1}^{k} \frac{Q_j}{\sigma^2} = \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2} \sim \chi^2(n) \longrightarrow (8)$$

we know that the sum of squares of n std normal variates follows a chisquare distn.

i.e. $x_i \sim N(0, \sigma^2)$

$$\frac{x_i}{\sigma^2} \sim N(0, 1)$$

$$\therefore \sum_{i=1}^{n} \frac{x_i^2}{\sigma^2} \sim \chi^2(n) \longrightarrow (9)$$

Using (8) and (9), we have, $\sum_{j=1}^{k} \dfrac{Q_j}{\sigma^2} = \dfrac{\sum_{i=1}^{n} x_i^2}{\sigma^2} \sim \chi^2(n) \longrightarrow (10)$

From (7) and (10), we can conclude that

$$\sum_{j=1}^{k} r_j = n.$$

Hence the proof.

Applications of Cochran's theorem:-

1) Using cochran's theorem, we can show that $\bar{x}$ and $s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1}$'s are independent.

2) Let $x_1, x_2, \ldots x_n$ be independent $N(0,1)$. Let us define

$$y_j = \sum_{j=1}^{n} x_j a_{ij} \quad ; \quad 1 \le j = n$$

where $a_{ij} = A$, is an orthogonal rotation with determinant of

$$|A| = 1 .$$

Then $\quad AA' = I = A'A .$